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UNIVERSITY OF CALIFORNIA AT BERKELEY

SOME NODAL NONCOMMUTATIVE JORDAN ALGEBRAS[1]

LOUIS A. KOKORIS

1. Introduction. An algebra A over a field F is called a noncommutative Jordan algebra if A is flexible and if the associated algebra A^+ is a Jordan algebra. In a recent study by Schafer [2] of noncommutative Jordan algebras of characteristic p, such an algebra A is called a nodal algebra if it is finite dimensional, has a unity element 1, and has the form $A=1F+N$ where every element of N is nilpotent but N is not a subalgebra of A. It is known [1] that nodal noncommutative Jordan algebras of characteristic zero cannot exist.

Since a noncommutative Jordan algebra is flexible, it satisfies

$$(x, y, z) + (z, y, x) = 0, \tag{1}$$

where $(x, y, z)=(xy)z-x(yz)$. This is the linearized form of the flexible law $(xy)x=x(yx)$. The associated algebra A^+ is obtained from the algebra A by redefining multiplication by $x\cdot y=(xy+yx)/2$. (When the characteristic of A is 2 we use $x\cdot y=xy+yx$.)

In this paper we give a construction for a class of nodal noncommutative Jordan algebras for every characteristic $p\neq 2$. A subclass of these algebras consists of simple algebras and a description of ideals is given for the algebras which are not simple. For the sake of completeness, we construct a nodal algebra of characteristic 2. These examples all have the stronger property that A^+ is an associative algebra.

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2. **Algebras of characteristic** $p\neq 2$. The construction is begun with a description of the algebra which will turn out to be A^+. Let A^+ be the associative, commutative free algebra over any field F of characteristic $p\neq 2$ generated by two elements x, y and with a unity element 1. Also, let $x^{p^m}=y^{p^n}=0$ for arbitrary positive integers m, n. Then A^+ has dimension p^{mn}. The product of two elements a, b in A^+ is indicated by $a\cdot b$. For convenience we write $x^0=y^0=1$. Thus every element of A^+ is a linear combination of elements of the form $x^a\cdot y^b$ with a, b nonnegative integers.

Now define the algebra A to be the same vector space as A^+ and with the bilinear multiplication defined by

$$\begin{aligned}&(x^a\cdot y^b)(x^c\cdot y^d)\\ (2)\qquad &= (ad-bc)(x^{a+c-1}\cdot y^{b+d-1}-x^{a+c}\cdot y^{b+d}+x^{a+c-1}\cdot y^{b+d-1}\cdot g)+x^{a+c}\cdot y^{b+d}\end{aligned}$$

where g is any nilpotent element of A. In order for (2) to make sense for all possible a, b, c, d, we make the convention that $x^a\cdot y^b=0$ whenever at least one of the exponents is negative. Observe that $A=1F+N$ where every element of N is nilpotent. That is, every element of A has the form $\alpha 1+z$ with α in F and z nilpotent. The product $xy=(x\cdot 1)(1\cdot y)=1+g$, g in N, so N is not a subalgebra of A. Characteristic p is essential because

$$0=(x^{p^m}\cdot y^b)(1\cdot y^d)=p^m d(x^{p^m-1}\cdot y^{b+d-1}+x^{p^m-1}\cdot y^{b+d-1}\cdot g).$$

Since A^+ is an associative algebra, A is Jordan admissible.

THEOREM 1. *The algebra A is flexible, and hence, A is a nodal noncommutative Jordan algebra.*

It is sufficient to show that the linear expression (1) holds for the elements $\alpha=x^a\cdot y^b$, $\beta=x^c\cdot y^d$, $\gamma=x^j\cdot y^k$. The work is simplified by noting that (2) implies $\beta\alpha=-\alpha\beta+2x^{a+c}\cdot y^{b+d}$ and $\beta\gamma=-\gamma\beta+2x^{c+j}\cdot y^{d+k}$. Then the left side of (1) with x, y, z replaced by α, β, γ becomes $2[(\alpha\beta)\cdot\gamma+(\gamma\beta)\cdot\alpha-\alpha(x^{c+j}\cdot y^{d+k})-\gamma(x^{a+c}\cdot y_b{}^{+d})]$. By (2), the expression in the brackets is

$$\begin{aligned}&[(ad-bc)+(jd-kc)]\\ &\quad\cdot(x^{a+c+j-1}\cdot y^{b+d+k-1}-x^{a+c+j}\cdot y^{b+d+k}+x^{a+c+j-1}\cdot y^{b+d+k-1}\cdot g)\\ &\quad+2x^{a+c+j}\cdot y^{b+d+k}-[a(d+k)-b(c+j)+j(b+d)-k(a+c)]\\ &\quad\cdot(x^{a+c+j-1}\cdot y^{b+d+k-1}-x^{a+c+j}\cdot y^{b+d+k}+x^{a+c+j-1}\cdot y^{b+d+k-1}\cdot g)\\ &\quad-2x^{a+c+j}\cdot y^{b+d+k}=0.\end{aligned}$$

This completes the proof of Theorem 1.

3. **Simple algebras and ideals.** The first result of this section is concerned with simple algebras.

THEOREM 2. *Let $x^p = y^p = 0$. Then the algebra A is simple.*

Suppose B is a nonzero ideal of A. Any element of B has the form $z = \sum_{a,b} \alpha(a, b)x^a \cdot y^b$. If $z \neq 0$, we look at the terms $x^a \cdot y^b$ involved in the expression of z with b minimal. We select the one of these which has minimal exponent a. Now consider zR_x^b where R_x denotes right multiplication by x. Formula (2) implies $zR_x^b = (-1)^b b! x^a$ plus terms involving higher powers of x or involving factors y. Then $zR_x^b R_y^a = (-1)^b b! a! 1$ plus an element in N. Since a and b are both less than p, z in B implies $1 + n$ is in B for some n in N. The element $1 + n$ has an inverse in A since n is nilpotent and A is power-associative [1, p. 473]. Thus 1 is in B and $B = A$.

THEOREM 3. *Let $x^{p^m} = y^{p^n} = 0$ and $m + n > 2$. Then the subalgebra B generated by the elements $x^a \cdot y^b$ with $a \geqq p^j$, $b \geqq p^k$, j, k any nonnegative integers less than m, n respectively, and such that $j + k > 0$, is a proper ideal of A.*

The proof follows immediately from (2).

4. **Characteristic 2.** A simple nodal (commutative) Jordan algebra of characteristic 2 is defined as follows. Let A be a 3 dimensional vector space over any field F of characteristic 2 with a basis 1, x, y. Define multiplication by taking 1 as the unity element, $xy = yx = 1$, $x^2 = y^2 = 0$. The general element z of A has the form $z = \alpha 1 + \beta x + \gamma y$ where α, β, γ are in F and $z^2 = \alpha^2$. This implies A is a (commutative) Jordan algebra since $z^2(wz) = \alpha^2 wz = (z^2 w)z$ for any z, w. Clearly $A = 1F + N$ where N is the 2 dimensional vector space with basis x, y and N is not a subalgebra since $xy = 1$. If B is a nonzero ideal with $z = \alpha 1 + \beta x + \gamma y \neq 0$ in B, then $zy = \alpha y + \beta 1$. If $\beta \neq 0$, zy has an inverse in A and this implies 1 is in B. If $\beta = 0$, $\alpha \neq 0$, then $(zy)x = \alpha 1$ is in B. Finally, if $\alpha = \beta = 0$, $zx = \gamma 1 \neq 0$ is in B. Thus $B = A$.

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