REMARKS ON A PAPER BY UTZ[1]

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In this note I shall study the behavior, as $t \to \infty$, of solutions of the equation

$$x'' + \alpha(x, x') + \beta(x, t) = 0. \tag{1}$$

I shall generalize some results of Utz [2].

THEOREM I. *If* $\alpha(x, y)$ *and* $\beta(x, y)$ *are real functions which satisfy the conditions*

$$\alpha(x, 0) = 0,$$
$$\beta(x, y) < 0 \text{ for } x < 0 \quad y > T,$$
$$\beta(x, y) > 0 \text{ for } x > 0 \quad y > T,$$

where T *is a constant, and* $x(t)$ *is a solution of* (1) *valid for all large values of* t, *then* x *is either oscillatory or ultimately monotone.*

A function is said to be oscillatory if and only if it has arbitrarily large zeros.

PROOF. Suppose that x does not oscillate; then x is of fixed sign for large t. We suppose that $x>0$ for large t (if $x<0$ for large t, a parallel argument holds). It is sufficient to show that x' is of fixed sign for large t. If $x'(t)=0$ for some value of t greater than T, then $x''(t) = -\beta[x(t), t]<0$. If $x'(t)=0$ for two values of t greater than T, then $x(t)$ has two maximum points with no minimum points between them. This is impossible; therefore x' is of fixed sign for large t, that is, x is ultimately monotone.

Birkhoff [1] shows that if $\alpha(x, 0)=0$, the sign of $\beta[x(t), t]$ is opposite to that of x for large t, and $x(t)$ and $x'(t)$ have the same sign for $t=t_0$, then $x(t)$ and $x'(t)$ have the same sign for $t>t_0$.

THEOREM II. *If*

(i) *all the assumptions made in Theorem* I *are satisfied,*

(ii) x *is ultimately monotone,*

(iii) $\limsup_{y\to\infty}\int_{a_1}^{y^t}\alpha[x(y), x'(y)]/x'(y)dy<\infty$

then x *is ultimately increasing or decreasing according as it is ultimately positive or negative.*

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In Utz's theorem [2] it is assumed that $\alpha(x, y)/y<0$; this assumption is stronger than (iii).

PROOF. It suffices to show that x'/x is ultimately positive. Let a_2 denote a number such that $x(t)$ is a solution of (1) valid for $t>a_2$, and let $a=\max(a_1, a_2)$. We define

$$P(t) = \exp \int_a^t \alpha[x(y), x'(y)]/x'(y)dy,$$

$$Q(t) = \beta[x(t), t]P(t)/x(t).$$

We may now write (1) in the form

$$\frac{d}{dt}[P(t)x'(t)] + Q(t)x(t) = 0.$$

We let

$$\theta(t) = \arctan [P(t)x'(t)/x(t)];$$

this is in effect the polar coordinate transformation in the xx' plane. We see that P and Q are positive when t is greater than a, and that P is bounded. Also

$$d\theta/dt = [(P'x' + Px'')/x - P(x'/x)^2] \cos^2 \theta$$

and since $P'(t)=\alpha[x(t), x'(t)]P(t)/x'(t)$, we have

$$\begin{aligned} d\theta/dt &= [(\alpha + x'')/x - (x'/x)^2]P \cos^2 \theta \\ &= [-\beta/x - (\tan \theta/P)^2]P \cos^2 \theta \qquad (2) \\ &= - Q \cos^2 \theta - \sin^2 \theta/P < 0 \text{ for } t > a. \end{aligned}$$

We now suppose that x'/x is negative for large t; then θ lies between $\pi/2$ and π for large t. Since θ is monotone decreasing for $t>a$, $\lim_{t\to\infty}\theta(t)$ exists; if we denote it by θ_0, $\pi/2 \leqq \theta_0 \leqq \pi$. Consequently $d\theta/dt$ vanishes for $\theta=\theta_0$. However, (2) shows that if $\pi/2 \leqq \theta_0 < \pi$, then $d\theta/dt<0$, for $r=\theta_0$. On the other hand if $\theta_0=\pi$, then θ is identically equal to π, and x' vanishes identically. This is impossible, and therefore x'/x is positive for large t. This completes the proof.

Birkhoff's argument [1] shows that if (i) and (iii) are satisfied, and $Q(t)$ as defined above, is bounded away from 0, then x' oscillates.

Utz's corollary in [2] can be generalized as follows:

THEOREM III. *Let $c(t)$ and $g(t)$ be differentiable functions such that*

$$c(t) > 0, \qquad c'(t) \geqq 0, \text{ for } t \geqq T.$$

$$g(0) = 0, \qquad g'(t) \leqq 0.$$

Then if x is a solution of the equation

$$x'' + g(x') + c(t)x = 0 \tag{3}$$

valid for all large values of t, then either x is oscillatory, or $\lim_{t\to\infty} x(t) = \infty$, *or* $\lim_{t\to\infty} x(t) = -\infty$.

PROOF. It is easily verified that all the assumptions made in Theorem II are satisfied by (3). Hence x is either oscillatory or it is ultimately monotone, and x'/x is ultimately positive.

If we set $x' = v$ and differentiate (3), we obtain

$$v'' + g'(v)v' + c(t)v + c'(t)x = 0. \tag{4}$$

If x is nonoscillatory, then it is of fixed sign for large t. Without loss in generality we may assume that $x > 0$ for large t. Then there exists a number T_1 such that x and v are positive for $t > T_1$. Whenever v' vanishes, $v'' = -cv - c'x$, and this quantity is negative if t exceeds max (T, T_1). Hence if v' vanishes for two values of t greater than max (T, T_1), then v has two maximum points with no minimum points between them. Since this is impossible, v' is of fixed sign for large t. Moreover v' cannot be negative for large t, as this would imply that v'' is negative for large t and hence that v tends to $-\infty$ as t tends to ∞ (cf. [2]). It follows that v' is positive for large t, that is, x'' is ultimately positive. Since x' is also ultimately positive, it follows that x tends to ∞ as t tends to ∞. By a similar argument it can be shown that if x is negative for large values of t then x tends to $-\infty$ as t tends to ∞.

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REFERENCES

1. G. Birkhoff, *Stability of spherical bubbles*, Quart. Appl. Math. vol. 13 (1956) pp. 451–453.

2. W. R. Utz, *A note on second-order nonlinear differential equations*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 1047–1048.

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