ON THE GIBBS' PHENOMENON IN A CERTAIN EIGENFUNCTION SERIES

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I. **Introduction.** In a paper by B. Friedman and Luna Mishoe [1] it is shown that the series,

$$\sum_{-\infty}^{\infty} a_n u_n(x); \tag{1}$$

where

$$a_n = \left(\int_0^1 u_n(\xi) B^* v_n(\xi) d\xi\right)^{-1} \int_0^1 f(\xi) B^* v_n(\xi) d\xi;\ u_n(x)$$

being eigenfunctions or nonzero solutions of the system

$$u'' + q(x)u + \lambda[p(x)u - u'] = 0, \qquad u(0) = u(1) = 0 \tag{2}$$

and $v_n(x)$ being eigenfunctions of the system adjoint to (2), where $B^* = d/dx + p(x)$; converges to;

$$1/2[f(x+) + f(x-)] + c \exp \int_0^x p dt; \tag{3}$$

provided $f(x)$ is of bounded variation, and $p(x)$ possesses a continuous second derivative, and $q(x)$ is continuous on (0, 1). In the expression (3) above

$$c = -1/2\left[f(0+) + f(1-) \exp - \int_0^1 p dt\right]. \tag{4}$$

In this paper the following theorem is proved:

THEOREM. *The partial sum* $\sigma_n(x) \equiv \sum_{k=-n}^{n} a_k u_k(x)$ *exhibits Gibbs' phenomenon for the function* $g(x) \equiv f(x) + c \exp \int_0^x p dt$ *whenever the Fourier partial sum* $S_n(x) \equiv \sum_{k=-n}^{n} \exp(2k\pi i x) \int_0^1 f(\xi) \exp(-2k\pi i \xi) d\xi$ *exhibits Gibbs' phenomenon for* $f(x)$.

II. **A relationship between** $\sigma_n(x)$ **and** $S_n(x)$. In [1] it is shown that:

$$u_n(x) = \lambda_n^{-1}\left[\exp\left(\lambda_n x - \int_0^x p dt\right) - \exp\left(\int_0^x p dt\right)\right] + O\left(\frac{1}{\lambda_n^2}\right), \tag{5}$$

$$B^* v_n(x) = \exp\left(-\lambda_n x + \int_0^x p dt\right) + O\frac{(e^{-\lambda_n x})}{\lambda_n}$$

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where $O(1)$ means a bounded function as $|\lambda| \to \infty$.

In [2] it is shown that:

$$\lambda_n = 2n\pi i + \int_0^1 p\,dt + O\left(\frac{1}{n}\right). \tag{6}$$

From (5) and (6) it follows that

$$\begin{aligned}\int_0^1 u_n(\xi)B^*v_n(\xi)d\xi &= \int_0^1 \frac{1}{\lambda}\,d\xi + \int_0^1 O\left(\frac{1}{\lambda}\right)d\xi \\ &\quad - \int_0^1 \exp\left(-\lambda_n\xi + \int_0^\xi p\,dt\right)d\xi = \frac{1 + O(1/\lambda_n)}{\lambda_n}\end{aligned} \tag{7}$$

and consequently

$$\left[\int_0^1 u_n(\xi)B^*v_n(\xi)d\xi\right]^{-1} = \frac{\lambda_n}{1 + O(1/n)} = \lambda_n\left[1 + O\left(\frac{1}{n}\right)\right] \tag{8}$$

for large n.

Now in (5) if we use the fact that $\exp\left(-\int_0^x p\,dt\right) = 1 + O(1)$ we have

$$u_n(x) = \lambda_n^{-1}\left[e^{\lambda_n x} + O(1) + O\left(\frac{1}{n}\right)\right] \tag{9}$$

and

$$\int_0^1 f(\xi)B^*v_n(\xi)d\xi = \int_0^1 f(\xi)e^{-\lambda_n\xi}d\xi + O(1). \tag{10}$$

From (8), (9), and (10) we obtain

$$\begin{aligned}a_n u_n(x) &\equiv u_n(x)\left[\int_0^1 u_n(\xi)B^*v_n(\xi)d\xi\right]^{-1}\int_0^1 f(\xi)B^*v_n(\xi)d\xi \\ &= e^{\lambda_n x}\int_0^1 f(\xi)e^{-\lambda_n\xi}d\xi + O(1).\end{aligned} \tag{11}$$

Using the value of λ_n as given in (6) we have

$$e^{\lambda_n x} = e^{2n\pi i x}\cdot\exp\left(2x\int_0^1 p\,dt + O\left(\frac{x}{n}\right)\right) = e^{2n\pi i x} + O(1) \tag{12}$$

and consequently

$$a_n u_n(x) = e^{2n\pi i x}\int_0^1 f(\xi)e^{-2n\pi i\xi}d\xi + E_n(x), \tag{13}$$

where

$$E_n(x) = O(1) + O(1/n).$$

And we have,

$$(14)\quad \sigma_n(x) \equiv \sum_{k=-n}^{n} a_k u_k(x) = \sum_{k=-n}^{n} e^{2k\pi ix} \int_0^1 f(\xi) e^{-2k\pi i\xi} d\xi + \sum_{k=-n}^{n} E_k(x)$$

or $\sigma_n(x) = S_n(x) + T_n(x)$ where $S_n(x)$ is the Fourier partial sum of $f(x)$ and $T_n(x) \equiv \sum_{k=-n}^{n} E_k(x)$. This is our desired relationship.

III. **Uniform convergence of the series $\sum_{-\infty}^{\infty} E_n(x)$ in (0, 1).** Since [1] $\sum_{-\infty}^{\infty} a_n u_n(x) = 1[f(x+) + f(x-)]/2 + c \exp \int_0^x p dt$; it follows that

$$(15)\quad \begin{aligned} &\sum_{-\infty}^{\infty} \left[e^{2n\pi ix} \int_0^1 f(\xi) e^{-2n\pi i\xi} d\xi + E_n(x) \right] \\ &\qquad = 1/2[f(x+) + f(x-)] + c \exp\left(\int_0^x p dt \right) \end{aligned}$$

and since

$$\sum e^{2n\pi ix} \int_0^1 f(\xi) e^{-2n\pi i\xi} d\xi = 1/2[f(x+) + f(x-)] \qquad \text{for } 0 < x < 1$$

we have

$$(16)\quad \begin{aligned} &\sum_{-\infty}^{\infty} \left[e^{2n\pi ix} \int_0^1 f(\xi) e^{-2n\pi i\xi} d\xi + E_n(x) \right] \\ &\qquad = c \exp\left(\int_0^x p dt \right) + \sum_{-\infty}^{\infty} e^{2n\pi ix} \int_0^1 f(\xi) e^{-2n\pi i\xi} d\xi \end{aligned}$$

and it follows that

$$(17)\quad \sum_{-\infty}^{\infty} E_n(x) = c \exp\left(\int_0^x p dt \right).$$

Now since $p(x)$ possess a continuous second derivative on (0, 1) it follows that $c \exp (\int_0^x p dt)$ is both continuous and of bounded variation on (0, 1) and therefore may be expanded into a Fourier series which converges uniformly to $c \exp (\int_0^x p dt)$ for $0 < x < 1$. From this we conclude that

$$(18)\quad \sum_{-\infty}^{\infty} E_n(x) = c \exp\left(\int_0^x p dt \right) \qquad \text{uniformly for } 0 < x < 1.$$

IV. **Gibbs' Phenomenon.** Let $f(x)$ possess a finite discontinuity at $x=a_1$ and let α_n and β_n be the abscissa and ordinate respectively of the first maxima (or minima) of $y=S_n(x)$ to the left or right of $x=a_1$, then as

$$n \to \infty, \qquad \alpha_n \to a_1. \tag{19}$$

But on the left of a_1

$$\beta_n \to f(a_1-0) + \left[\frac{f(a_1+0)-f(a_1-0)}{\pi}\int_\pi^\infty \frac{\sin x}{x}\,dx\right]$$

and on the right of $x=a_1$

$$\beta_n \to f(a_1+0) - \left[\frac{f(a_1+0)-f(a_1-0)}{\pi}\int_\pi^\infty \frac{\sin x}{x}\,dx\right].$$

This is Gibbs' Phenomenon for the partial sum $S_n(x)$ of the Fourier Series of $f(x)$.

Since $T_n(x)=c\exp\left(\int_0^x p\,dt\right)$ uniformly in $(0, 1)$ as $n\to\infty$, this partial sum does not exhibit Gibbs' Phenomenon.

Note that corresponding to the maxima (or minima) (α_n, β_n) on the curve, $y=S_n(x)$ there is a point (α_n, β_n^1) on the curve $y^1=\sigma_n(x)$ such that

$$\beta_n^1 = \beta_n + c\exp\left(\int_0^{\alpha_n} p\,dt\right).$$

Now as $n\to\infty$, $\alpha_n\to a_1$

$$\beta_n^1 \to f(a_1-0) + \left[\frac{f(a_1+0)-f(a_1-0)}{\pi}\int_\pi^\infty \frac{\sin x}{x}\,dx\right] + c\exp\left(\int_0^{a_1} p\,dt\right)$$

on the left of $x=a_1$ and

$$\beta_n^1 \to f(a_1+0) - \left[\frac{f(a_1+0)-f(a_1-0)}{\pi}\int_\pi^\infty \frac{\sin x}{x}\,dx\right] + c\exp\left(\int_0^{a_1} p\,dt\right)$$

on the right of $x=a_1$. Or as $n\to\infty$

$$\alpha_n \to a_1; \tag{20}$$

and since $f(a_1+0)-f(a_1-0)=g(a_1+0)-g(a_1-0)$ where

$$g(x) = f(x) + c \exp\left(\int_0^x p dt\right)$$

we have

$$\beta_n^1 \to g(a_1 - 0) + \left[\frac{g(a_1 + 0) - g(a_1 - 0)}{\pi} \int_\pi^\infty \frac{\sin x}{x} dx\right] \tag{21}$$

on the left of $x=a_1$ and

$$\beta_n^1 \to g(a + 0) - \left[\frac{g(a_1 + 0) - g(a_1 - 0)}{\pi} \int_\pi^\infty \frac{\sin x}{x} dx\right] \tag{22}$$

on the right of $x=a_1$, where $g(x)=f(x)+c \exp (\int_0^x p dt)$. The results in (20), (21), and (22) shows clearly that $y^1=\sigma_n(x)$ exhibits Gibbs' Phenomenon for

$$g(x) = f(x)c \exp\left(\int_0^x p dt\right)$$

wherever $y=S_n(x)$ exhibits this Phenomenon for $f(x)$.

References

1. Bernard Friedman and L. I. Mishoe, *Eigenfunction expansions associated with a non-self-adjoint differential equation*, Pacific J. Math. vol. 6, no. 2 (1956).

2. L. I. Mishoe and G. C. Ford, *On the uniform convergence of a certain eigenunction series*, Pacific J. Math. vol. 6, no. 2 (1956).

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