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CUT SETS IN TOTALLY NONAPOSYNDETIC CONTINUA[1]

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For metric spaces, F. Burton Jones has shown [**1**, Theorem 12] that if D is an open subset of a compact continuum M such that M is nonaposyndetic at every point of D, then D contains a point x and M contains points y and z such that y cuts x from z in M. This paper extends the results of Jones to a certain class of topological spaces, gives a cut set in some cases where there is no cut points and gives stronger cutting properties.

DEFINITION.[2] If A and B are two mutually exclusive subsets of a topological continuum (i.e., a connected topological space) T and there does not exist a continuum T' and an open subset U of T such that $[T-A]\supset T'\supset U\supset B$, then T is *nonaposyndetic at B with respect to A*.

DEFINITION. If T is a topological continuum, A is a subset of T and G is a collection of subsets of T and if, for each point x of A, there is a member g of G such that T is nonaposyndetic at x with respect to g, then T is *totally nonaposyndetic on A with respect to G.* If $A=T$ then T is *totally nonaposyndetic with respect to G.* If, in addition, $G=\{\{x\}\mid x \text{ is in } T\}$ then T is *totally nonaposyndetic.*

DEFINITION. If T is a topological continuum, Z is the smallest cardinal number of a topological basis for T and the subset A of T

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[2] The first two definitions are generalizations of definitions due to Jones [**1**]. Terms not defined herein are used as in [**1**] or [**3**].

is the nonvacuous common part of Z open subsets of T, then A is a *domain intersection* sub*set* of T. If A is the nonvacuous common part of Z dense, open subsets of T, then A is a *dense-domain intersection* sub*set* of T

The reader should note that a domain intersection subset of a perfectly separable topological continuum is an inner limiting set (G_δ set) relative to that continuum.

DEFINITION. If (1) Z is a cardinal number, (2) G is a collection of proper subsets of a topological continuum T, and (3) E is a collection of open sets the cardinality of which is Z if Z is transfinite and finite if Z is finite, such that, if g is a member of G and T_1 is a continuum in $T-g$, then there is a member E' of E such that $T-T_1 \supset E' \supset g$; then G is said to have the *Z-domain property* (relative to T) and E is said to be a *Z-domain collection* for G (relative to T).

It should be noted that if G is a collection of nondense subsets of a topological continuum T which has a basis of cardinality Z, then G has the Z-domain property if either Z is finite or each member of G is a bicompact set.

DEFINITION.[3] If T is a topological continuum and there is a topological basis for T of cardinality Z' such that T is not the sum of Z', or less than Z', closed subsets of T, each nowhere dense in T, then T is a *Baire topological* continuum. If, in addition, D is an open subset of T and D is not contained in the sum of Z', or less than Z', closed subsets of T, each nowhere dense in T, then T is *Baire topological on D*. If, furthermore, x is a point of T and for each open subset U of T containing x, the continuum T is Baire topological on U, then T is *Baire topological at x*. If T is Baire topological at each point of a subset A of T, then T is said to be *locally Baire topological on A*.

A complete, perfectly separable, metric continuum is a locally Baire topological continuum.

Standing notation. The point set T is a Baire topological continuum. If A is a subset of T then $K(A)$ is the set of all points x of T such that x is in A or T is nonaposyndetic at x with respect to A. The smallest cardinal number of a basis for T is Z. The first ordinal number that is preceded by Z ordinals is z. The collection G of proper subsets of T has the Z-domain property and E is a Z-domain collection for G. The cardinality of E is W. The point set P is a subset of T of cardinality V such that if g is in G then $P \cdot [T-g] \neq 0$. The sequence $p(1)$, $p(2), \cdots, p(i), \cdots$, is a most economical well-ordering of P. If A

[3] Compare with the properties, for a continuum, of being of the second category in itself and of being of the second category in itself at each point.

is a subset of T which does not contain P, then $F(A)$ is the first term of P in $T-A$.

LEMMA. *No open subset of a topological continuum is contained in the sum of a finite number of closed sets, each nowhere dense in the continuum.*

THEOREM 1. *If T is Baire topological on the open set D and I is a dense-domain intersection subset of T, then the closure $\bar{I}$ of I contains an open subset of D.*

PROOF. Assume $\bar{I}$ does not contain an open subset of D. Then $\bar{D}\cdot\bar{I}$ is a nowhere dense closed set. The complement of I is contained in the sum of Z or less than Z closed, nowhere dense sets and $\bar{D}\cdot\bar{I} \supset D\cdot I$. Therefore D is contained in the sum of $Z+1$ or less than $Z+1$ closed, nowhere dense sets. By the lemma, $Z+1$ cannot be finite. Hence $Z+1=Z$, and D is contained in the sum of Z or less than Z closed, nowhere dense sets. This contradicts the choice of D. Therefore $\bar{I}$ contains an open subset of D

THEOREM 2. *T contains a dense-domain intersection subset I such that if x is a point of I and T is nonaposyndetic at x with respect to the set g of G, then g cuts x from $F(g)$ in T.*

PROOF. Either $W=Z$ or W is finite. In either case T is not the sum of W closed, nowhere dense subsets of T. Let w be the first ordinal that is preceded by W ordinals. Let $E_1, E_2, \cdots, E_i, \cdots$, for $i<w$, be a most economical well-ordering of E. For $i<w$, let H_i be the set of all points x in T such that either (1) x is in E_i or (2) x is not in $K(E_i)$ or (3) x is not in the $F(E_i)$-component of $T-E_i$.

The set H_i is a dense open subset of T. It is open since it is defined as the sum of open sets. If it were not dense in T, there would be a point y in $[T-E_i]\cdot K(E_i)$ and an open subset u of T such that $[T-E_i]\supset$ [the $F(E_i)$-component of $T-E_i]\supset u\supset y$. But this contradicts the definition of $K(E_i)$.

Let $I=\prod_{i<w} H_i$. The set I is a dense-domain intersection subset of T. Assume there is a point x of I and a member g of G such that T is nonaposyndetic at x with respect to g and g does not cut x from $F(g)$ in T. Then there is a continuum H such that $[T-g]\supset H \supset [x+F(g)]$. There is an open set E_j in E such that $[T-H]\supset E_j\supset g$. But T is nonaposyndetic at x with respect to g and, consequently, T is nonaposyndetic at x with respect to E_j. Also, $E_j\cdot F(g)=0$ and so $F(g)=F(E_j)$. Therefore, x is in the $F(E_j)$-component of $T-E_j$. But this means x is not in H_j and consequently not in I. This is a contradiction. Therefore if x is in I and T is nonaposyndetic at x

with respect to the set g in G, then g cuts x from $F(g)$ in T.

THEOREM 3. *The intersection of Z or less than Z dense-domain intersection subsets of T is a dense-domain intersection set.*

If Z is finite, Theorem 3 follows from the lemma. If Z is transfinite, Theorem 3 follows from the fact that the sum of Z or less than Z collections, each containing Z or less than Z elements, contains not more than Z elements.

THEOREM 4. *If D is an open set and J is a dense-domain intersection subset of T such that T is Baire topological on D and T is totally nonaposyndetic on $D \cdot J$ with respect to G, then there is a dense-domain intersection set I contained in J such that $\bar{I}$ contains some open subset of D and if x is in I and T is nonaposyndetic at x with respect to g of G, then g cuts x from $F(g)$ in T.*

PROOF. By Theorem 2, there is a dense domain intersection set I' such that if x is in I' and T is nonaposyndetic at x with respect to g of G, then g cuts x from $F(g)$ in T. By Theorem 3 (if $Z=1$, then $I'=J$), the intersection I of J and I' is a dense-domain intersection subset of T. From Theorem 1 it follows that $\bar{I}$ contains some open subset of D.

THEOREM 5. *If the cardinality of P is not greater than Z, then T contains a dense-domain intersection set I such that if x is a point of I and T is nonaposyndetic at x with respect to g of G, then g cuts x from each point of $(P+g)-g$ in T.*

PROOF. The ordinal number v is the first ordinal number that is preceded by V ordinals. If $i<v$ and $j<v$, $i\neq 1$ and $i\neq j$, then let $p_j(i)=p(i)$, $p_j(j)=p(1)$ and $p_j(1)=p(j)$. If A is a subset of T that does not contain P, then let $F_j(A)$ be the first term of the series $p_j(1), p_j(2), \cdots, p_j(k), \cdots$, for $k<v$, which is in $T-A$.

By Theorem 2, there is a dense-domain intersection subset I_j of T such that if x is in I_j and T is nonaposyndetic at x with respect to g of G then g cuts x from $F_j(g)$ in T. By Theorem 3, the set $I=\prod_{j<v} I_j$ is a dense-domain intersection subset of T.

If x is in I and T is nonaposyndetic at x with respect to g of G, then g cuts x from each point of $(P+g)-g$. Otherwise, if j were an ordinal such that $p(j)\cdot g=0$ and g did not cut x from $p(j)$, then, since $p(j)=p_j(1)=F_j(g)$, the set g would not cut x from $F_j(g)$ and, consequently, x would not belong to I_j.

THEOREM 6. *If the cardinality of P is not greater than Z, the set D is open, and J is a dense-domain intersection subset of T such that T is*

Baire topological on D and T is totally nonaposyndetic on $D \cdot J$ with respect to G; then there is a dense-domain intersection set I contained in J such that $\bar{I}$ contains some open subset of D and if x is in I and T is nonaposyndetic at x with respect to g of G, then g cuts x from each point of $(P+g)-g$ in T.

Theorem 6 follows from Theorem 4 by a construction similar to that in the proof of Theorem 5.

THEOREM 7 (THEOREM 8). *Under the hypothesis of Theorem* 4 (*Theorem* 6), *if T is locally Baire topological on D, then the closure of I, in the conclusion of Theorem* 4 (*Theorem* 6), *contains D.*

COROLLARY 1. *If* (1) *M is a perfectly separable, complete, metric continuum,* (2) *D is an open set such that M is nonaposyndetic at each point of D, and* (3) *Q is a countable subset of M, then D contains an inner limiting subset I of M, dense in D, such that if x is in I and M is nonaposyndetic at x with respect to the point y, then y cuts x from each point of $(Q+y)-y$ in M.*

COROLLARY 2. *If M is a totally nonaposyndetic, compact, metric continuum, then M contains a weak cut point.*

COROLLARY 3. *If M is a compact, metric continuum which is totally nonaposyndetic with respect to some collection H of closed sets, then some member of H cuts M* (*weakly*).

This theory gives information about cut points in finite, connected spaces.

EXAMPLE 1. Let T be the connected topological space consisting of the three points 0, 1, and 2 and the three regions $\{2\}$, $\{1, 2\}$, and $\{0, 2\}$. Let G consist of the sets $\{0\}$, $\{1\}$, and $\{2\}$. Let E be the collection of regions. E is a minimal basis for T. Let $P=T$. Each member of G consists of one point, so G has the 3-domain property, E is a 3-domain collection for G, and P has a point in the complement of each of the members of G. The continuum T is nonaposyndetic at each point with respect to each member of G not containing the point. Since T has a finite basis, T is a locally Baire topological continuum. Hence, the hypothesis of Theorem 8 is satisfied and therefore T contains a dense-domain intersection subset I, dense in T, such that if x is in I and T is nonaposyndetic at x with respect to g of G then g cuts x from each point of $T-g$. But 2 is not in the closure of $T-2$ so 2 is in I. In addition, T is nonaposyndetic at 2 with respect to $\{0\}$ and $\{1\}$ of G. Therefore 0 cuts 2 from 2 and from 1 and 1 cuts 2 from 2 and from 0.

That this theory sometimes gives a cut set when there is no cut point is illustrated by the continuum T described in Example 2 and indicated by Figure 1.

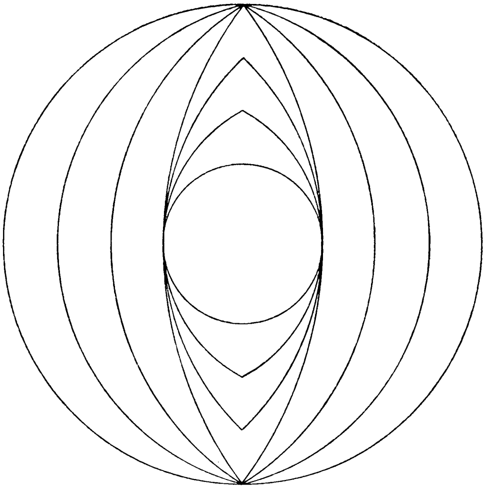

FIG. 1

EXAMPLE 2. Let C be the set of numbers in the interval from 0 to 1 that can be written to the base 3 using only the digits 0 and 2. Let C' be the set of all numbers of the form $x-1/2$, for x in C. T is the sum of the following arcs of circles: (1) for each x in C', the arcs from $(0, x)$ to $(0, -x)$ through $(1/6, 0)$ and through $(-1/6, 0)$; and (2) for each x in C', the arc from $(0, 1/2)$ to $(0, -1/2)$ through $(x, 0)$.

T is aposyndetic, in fact each closed half plane with $(0, 0)$ in it intersects T in a continuum, and T does not contain a weak cut point. However, T is totally nonaposyndetic with respect to the collection of pairs of points of T; in fact, if x is a point of T, then either T is nonaposyndetic at x with respect to $\{(0, 1/2), (0, -1/2)\}$ or T is nonaposyndetic at x with respect to $\{(1/6, 0), (-1/6, 0)\}$. But T is a compact metric continuum. Therefore, by Corollary 3, some pair of points of T cuts T.

The theorems of this paper are strongly dependent on the Baire topological condition.

EXAMPLE 3. There is a connected topological space T' which is totally nonaposyndetic but which does not contain a weak cut point.

Let a_0 and a_1 be two points of three dimensional Euclidean space. Let A_{11}, A_{12}, and A_{13} be three mutually exclusive arc segments from

a_0 to a_1. Let $A_1 = a_0 + a_1 + \sum_{i=1}^{3} A_{1i}$. Let a_{11}, a_{12} and a_{13} be points of A_{11}, A_{12} and A_{13} respectively. If n is a positive integer, let a_{n+1} be a point not in $\sum_{i=1}^{n} A_i$. For $i=1, 2, 3$, let $A_{(n+1)i}$ be an arc segment from a_{ni} to a_{n+1} in the complement of $\sum_{i=1}^{n} A_i$ and let $a_{(n+1)i}$ be a point in $A_{(n+1)i}$ such that $A_{(n+1)1}$, $A_{(n+1)2}$, and $A_{(n+1)3}$ are mutually exclusive and each point of $\sum_{i=1}^{n} A_i$ is within the distance $1/n$ of some point of $A_{(n+1)i}$ between a_{ni} and $a_{(n+1)i}$. Let $A_{n+1} = a_{n+1} + \sum_{i=1}^{3} [a_{ni} + A_{(n+1)i}]$ and let $T' = \sum A_i$. The space T' is a connected subspace of a perfectly separable, metric space so T' is a connected topological space. Not only is T' totally nonaposyndetic, but T' is nonaposyndetic at each point of T' with respect to each other point of T''. Each point of a Baire topological continuum having these properties is a weak cut point of the continuum. However, the perfectly separable, metric space T' contains no weak cut point. In fact, if x is a point of T' then $T' - x$ is arcwise connected.

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