REGULAR PARTIAL DIFFERENTIAL EQUATIONS

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1. **Introduction.** We consider below the Cauchy problem for systems of homogeneous linear partial differential equations (DE's) with constant coefficients. That is, we ask for solutions of

$$(1) \qquad \partial u_j/\partial t = \sum p_{jk}(D_1, \cdots, D_r)u_k \qquad [j, k = 1, \cdots, n],$$

satisfying $\boldsymbol{u}(\mathrm{x}; 0) = \boldsymbol{v}_0(\mathrm{x})$, where $\boldsymbol{v}_0(\mathrm{x})$ is supposed given in infinite $(x_1, \cdots, x_r)$-space. Here $\|p_{jk}(\boldsymbol{D})\|$ denotes a general matrix of polynomials with constant coefficients in the differential operators $D_i = \partial/\partial x_i$.

Our main result is an existence theorem for the Cauchy problem in *infinite space*, for any system (1) which is *regular* in the following sense [**1**]. For each real wave-vector $\boldsymbol{q}$, we let $\lambda_1(\boldsymbol{q}), \cdots, \lambda_n(\boldsymbol{q})$ denote the characteristic values of the $\boldsymbol{q}$-matrix $\|p_{jk}(i\boldsymbol{q})\|$, $i = (-1)^{1/2}$. We define the *spectral norm* of (1) as

$$(2) \qquad \sigma[P] = \sup_{\boldsymbol{q}, j} \operatorname{Re}\{\lambda_j(\boldsymbol{q})\} = \sup_{\boldsymbol{q}} \sigma[P(\boldsymbol{q})],$$

and call the system (1) *regular* when $\sigma[P] < +\infty$.

In this existence theorem, the solutions are defined as the semi-orbits, for $t > 0$, of a *C_0-semigroup* acting on an appropriate Banach space. Thus, in particular, we solve a family of "abstract Cauchy problems" ([**5**; **6**]), each corresponding in some sense to the system (1). Further, it is shown that the C_0-semigroup can be so constructed that all semi-orbits represent literal solutions of (1). Finally, in the "hyperbolic" case, a C_0-group can be constructed.

Though similar methods have been used by Gelfand and Šilov ([**3**; **4**]), the representation in terms of C_0-semigroups is new. The specific results of Hille [**5**] are for quite special DE's. Related results have also been obtained by L. Schwartz [**7**] and, using other methods, by L. Gårding (Acta Math. vol. 85 (1951) pp. 1–62) and A. Lax (Communications on Pure and Applied Mathematics, vol. 9 (1956) pp. 135–70).

2. **The Fréchet space Φ.** For any fixed wave-vector $\boldsymbol{q} = (q_1, \cdots, q_r)$, any system (1) has solutions of the special form

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$$u_j(\mathrm{x}; t) = f_j(q; t)e^{iq\cdot \mathrm{x}}, \qquad q\cdot \mathrm{x} = q_1x_1 + \cdots + q_rx_r. \tag{3}$$

Namely, it is sufficient that $f(q; t)$ satisfy

$$df_j/dt = \sum p_{jk}(iq)f_k \qquad [j, k = 1, \cdots, n], \tag{3*}$$

which may be regarded as a system of *ordinary DE*'s because q is fixed.

Now let Φ denote the vector space of all Borel functions $\phi = \phi(q)$, two such functions being identified when they differ on a set of measure zero. If we write formally

$$v(\mathrm{x}) = \int_Q e^{iq\cdot \mathrm{x}}\phi(q)dQ, \qquad dQ = dq_1 \cdots dq_r, \tag{4}$$

then (1) defines, through (3)–(3*), the group

$$T_t[\phi(q)] = \phi(q)e^{tP(iq)}, \qquad P(iq) = \|p_{jk}(iq)\|. \tag{5}$$

This is a symbolic solution of (1) in the Fourier transform space Φ dual to the space of $v(\mathrm{x})$.

Further, we can define Φ as a Fréchet space,[2] or topological linear space, by making

$$\phi_n \to \phi \quad \text{mean} \quad \phi_n(q) \to \phi(q) \text{ a.e.} \tag{6}$$

However, this formal solution is unsatisfactory, as a consideration of the case $u_t = \pm u_{xx}$ shows.

3. **Norms on Φ.** Various norms can be defined on Φ, as follows. Let $C = C(q)$ be any Borel function[3] from the space Q of real r-vectors q, to the space of $n \times n$ nonsingular complex matrices C. We define

$$N^2(q; \phi) = (\phi C, \phi C) = \phi C(q)C'^*(q)\phi'^* \tag{7}$$

as a complex inner product, and correspondingly

$$N(q; \phi) = [\phi C(q)C'^*(q)\phi'^*]^{1/2} \geqq 0. \tag{7'}$$

Since C is nonsingular, $N(q; \phi) = 0$ if and only if $\phi = \mathbf{0}$ in Φ (i.e., $\phi_j(q) = 0$ a.e.).

For any such $C(q)$, the "norm" $N(q, \phi)$ is a non-negative real-valued Borel function on $E_n \times Q$ which, for each real r-vector q, makes the space of complex n-vectors ϕ into a unitary space $E_n(N, q)$.

[2] C. Kuratowski, *Topologie*, first ed., p. 77.

[3] Function "measurable B" in the sense of C. Kuratowski, *Topologie*, 2d ed, vol. 1, p. 280. We require that the inverse image of a Borel set be a Borel set. Borel functions are measurable, and have the merit of being closed under composition (any Borel function of a Borel function is a Borel function).

LEMMA 1. *For any* $\phi \in \Phi$, *the integral*

$$N(\phi) = \int_Q N(q; \phi(q)) dQ \tag{8}$$

is defined, as a non-negative real number or $+\infty$.

PROOF. First, $N(q; \phi(q))$ is a Borel function for all $\phi \in \Phi$. Since $N(q; \phi) \geqq 0$, the conclusion follows immediately. In this proof, it is essential that the functions involved be Borel functions, and not merely measurable.

LEMMA 2. *Under the norm* (8), *the set of* $\phi \in \Phi$ *with* $N(\phi) < +\infty$ *is a Banach space* $B(N)$.

This result is an immediate corollary of standard Lebesgue theory, and the definition of a Banach space. Similar constructions have in fact been used by other authors.[4]

In the subsequent part of this paper, we shall adopt the following notational conventions. E_n will denote the unitary space of complex n-tuples $\phi = (c_1, \cdots, c_n)$ with norm $|\phi| = (\sum_{i=1}^n c_i \bar{c}_i)^{1/2}$. $E_n(N, q)$ will denote an r-parameter family of unitary spaces of complex n-tuples ϕ with norms $N(q; \phi)$. $B(N)$ will denote the Banach space of complex vector-valued functions ϕ with $N(\phi) = \int N(q; \phi) dQ$.

The norm or "modulus" of a linear operator $A(q)$ on $E_n(N, q)$ will be defined by $\|A(q)\|_N = \sup N(q; A(q)[\phi])$ for $N(q; \phi) = 1$. The norm of a bounded linear operator A on $B(N)$ will be defined by $\|A\|_N = \sup N(A[\phi])$ for $N(\phi) = 1$.

THEOREM 1. *For each* q, *let* $T_t(q)$ *be a semigroup of linear transformations on the space* E_n, *depending continuously on* q *and* $t \geqq 0$. *As operators on* $E_n(N, q)$, *let the* $T_t(q)$ *have uniformly bounded moduli on* $0 < t < 1$. *Then the semigroup* $\{T_t\}$ *on* $B(N)$, *defined by* (5) *and* (8) *for* $t > 0$, *is a* C_0-*semigroup*.

PROOF. Because of the continuous dependence of $T_t(q)$ on q, $T_t[\phi]$ is always a Borel function; hence $\{T_t\}$ is *always a semigroup* on Φ. If, for fixed t, the $T_t(q)$ have norms bounded by $K(t) < +\infty$ then, substituting (8) into (5), we get

$$N(T_t[\phi]) = K(t) \int_Q N(q; \phi(q)) dQ = K(t) \|\phi\|_N. \tag{9}$$

[4] F. Cunningham, *L*¹-*structure in Banach spaces*, Harvard Doctoral Thesis, 1953. For $p = 2$, see J. von Neumann, *On rings of operators, Reduction theory*, Ann. Math. vol. 50 (1949).

Hence T_t transforms the subspace $B(N)$ of Φ into itself, and is a linear operator of norm at most $K(t)$ on this Banach space.

It remains [6, p. 18] to show that, for each $\phi \in B(N)$ the orbit $T_t[\phi]$ tends continuously to ϕ as $t \downarrow 0$—i.e., that

$$0 = \lim_{t \to 0^+} N(T_t[\phi] - \phi) = \lim_{t \to 0^+} \int_Q N(q; T_t[\phi] - \phi) dQ.$$

But now, for any $\phi \in B(N)$,

$$0 \leqq N(q; T_t[\phi(q)] - \phi(q)) \leqq (K(t) + 1) N(q; \phi(q)),$$

where

$$\int N(q; \phi(q)) dQ = \|\phi\|_N < +\infty.$$

Hence Lebesgue's Dominated Convergence Theorem[5] applies, and, since $T_t[\phi] \to \phi(q)$ for each q (i.e., $N(q; T_t[\phi] - \phi) \to 0$ our result is proved.

We note that T_t is a direct integral[6] of operators on the subspaces of $B(N)$ corresponding to Borel subsets of Q; by (8) and (9), the *norm* of each T_t is the l.u.b. of the norms of the linear operators

$$\{\exp t \| p_{jk}(iq) \| \},$$

acting on the $E_n(N; q)$.

4. **Regular case.** We now assume that the system (1) is regular in the sense of (2), that $\sigma[P] < +\infty$. In this case, for each $q \in Q$, we can choose $C(q)$ so that, under the norm (7′), the modulus of $\{T_t(q)\}$ exceeds $e^{\sigma[P]t}$ by arbitrarily little.

LEMMA 3. *Given $q \in Q$ and $\eta > 0$, we can so choose $C(q)$ that, on $E_n(N; q)$,*

$$\|T_t(q)\|_N \leqq \exp\{(\sigma[P(q)] + \eta)t\}, \qquad t > 0. \tag{10}$$

PROOF. A slight modification of standard[7] arguments shows that there always exists a matrix $C(q; \eta)$ for $P(iq)$, such that

$$C^{-1} P(iq) C = J(q; \eta) \tag{11}$$

has entries $\lambda_j(q)$ on the main diagonal, zeros and η's just above this diagonal, and all other entries zero. For any such $C(q; \eta)$, the modulus $\|T_t(q)\|_N$ of $T_t(q)$ for the norm (7′) is that of e^{Jt} operating on the

[5] For this result, see E. J. McShane, *Integration*, Princeton, 1944, p. 168.

[6] L. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953, p. 176.

[7] See for example G. Birkhoff and S. MacLane, *A survey of modern algebra*, rev. ed., Ch. X.

unitary space of complex n-vectors ψ, under the ordinary norm $(\psi\psi'^*)^{1/2}=|\psi|$. But an explicit calculation shows that this is at most $\exp\{(\sigma[P(q)]+\eta)t\}$.

COROLLARY. *Let $q_0\in Q$ be given, and let $\sigma>\sigma(P[q_0])$. Then $C=C(q_0)$ exists such that, throughout some neighborhood $R(q_0)$ of q_0,*

$$\|T_t(q)\|_N \leqq e^{\sigma t} \qquad \textit{for all } t>0. \tag{12}$$

This follows from Lemma 3, because of the continuity of the modulus of $\|T_t(q)\|_N$, as a function of q, for any fixed $C_0=C(q_0)$.

5. **Abstract Cauchy problem.** The preceding results lead directly to a solution of the abstract Cauchy problem, for any regular system (1). We first prove

THEOREM 2. *Let* (1) *be regular, and let $\sigma>\sigma(q)$ for all $q\in Q$. Then a Borel function $C(q)$ exists, such that* (12) *holds for all $q\in Q$.*

PROOF. The $R(q_0)$ defined by the corollary of Lemma 3 constitute an open covering of Q. But, by Lindelöf's Theorem,[8] one can select from any open covering a countable subcovering $R_1, R_2, R_3, \cdots$. Replacing each R_h by $Q_h=R_h\cap(R_1\cup\cdots\cup R_{h-1})'$, we get finally a partition of Q into countably many Borel sets Q_h, such that (12) holds for all $q\in Q$.

Combining Theorems 1 and 2, we get the following solution to an abstract Cauchy problem corresponding to (1) in a Fourier transform space.

COROLLARY. *Let* (1) *be regular. Then there exists a Borel function $C(q)$, such that the associated Banach space $B(N)$ in Φ admits a C_0-semigroup, whose semi-orbits satisfy* (3*), *with $\|T_t\|_N\leqq e^{\sigma t}$ for $t\geqq 0$.*

6. **Differentiable functions.** We now show that, for $\phi_0(q)$ vanishing fast enough at infinity, the semi-orbits of $T_t[\phi_0]$ for $t\geqq 0$ define, through (4), solutions of (1) in the literal sense.[9]

It is well known [2, p. 8] that $v(x)$ is defined by (4) as a continuous function, whenever $\phi(q)\in L_1$. The following result is also easily established.

LEMMA 4. *Let $\phi\in\Phi$ satisfy*

$$\int |\phi(q)|\,(1+\sum |p_{jk}(iq)|)dQ<+\infty, \tag{13}$$

[8] See G. T. Whyburn, *Analytic topology*, New York, 1942, p. 4.

[9] This is closely related to some results of [3] and [4]; see also [2, pp. 26, 57, 125] and [7].

where $|\phi| = (\phi\phi'^*)^{1/2}$ *is the ordinary norm on* E_n. *Then, if* $v(x)$ *is defined by* (4),

$$\sum p_{jk}(D)v_k(x) = P_j[v] = \int_Q \exp(i\mathbf{q}\cdot \mathrm{x} \sum p_{jk}(i\mathbf{q})\phi_k(\mathbf{q})dQ) \tag{14}$$

exists for all real r-vectors $\mathbf{x}$.

Proof. By (13), the right side of (14), whose integrand is a Borel function[3] and so measurable, is Lebesgue integrable to a continuous function. The rest of the proof is an obvious extension of standard results [**2**, p. 8].

The following sharper result is less easy.

Theorem 3. *For all* τ, $|\tau - t| < \epsilon$, *let* $\mathbf{f}(\mathbf{q}; t)$ *satisfy* (3*), *and also*

$$|\mathbf{f}(\mathbf{q};\tau)|(1 + \sum |p_{jk}(i\mathbf{q})|) \leqq M(\mathbf{q}), \qquad a.e., \tag{13*}$$

where

$$\int M(\mathbf{q})dQ < +\infty.$$

Then (1) *is satisfied by the Fourier transform*

$$\mathbf{u}(\mathrm{x}; t) = \int \mathbf{f}(\mathbf{q}; t)e^{i\mathbf{q}\cdot\mathrm{x}}dQ. \tag{15}$$

Proof. By Lemma 4, the right side of (1) is defined as a continuous function in space, throughout $|\tau - t| < \epsilon$. Therefore, if $|\Delta t| < \epsilon$, the difference quotient $\Delta\mathbf{u}/\Delta t$ is defined. Hence, if (1) failed, we could find an $\mathbf{x}$ and a sequence of $\Delta t_m \to 0$, such that

$$(\Delta\mathbf{u}/\Delta t)_m = [\mathbf{u}(\mathrm{x}; t + \Delta t_m) - \mathbf{u}(\mathrm{x}; t)]/\Delta t_m$$

failed to converge to $\mathbf{P}[\mathbf{u}(\mathrm{x}; t)]$, as defined by (1). We shall now show that this is impossible, which will complete the proof. Indeed, for each $\mathbf{x}$

$$\frac{\Delta\mathbf{u}}{\Delta t_m} = \frac{1}{\Delta t_m}\int e^{i\mathbf{q}\cdot\mathrm{x}}[\mathbf{f}(\mathbf{q}; t + \Delta t_m) - \mathbf{f}(\mathbf{q}; t)]dQ$$

$$= \int e^{i\mathbf{q}\cdot\mathrm{x}}dQ\left\{\frac{1}{\Delta t_m}\int_t^{t+\Delta t_m} \mathbf{P}[\mathbf{f}(\mathbf{q};\tau)]d\tau\right\},$$

where $\mathbf{P}[\mathbf{f}]$ is defined by $P_j[\mathbf{f}] = \sum p_{jk}(i\mathbf{q})f_k$. Hence

$$\frac{\Delta\mathbf{u}}{\Delta t_m} - \mathbf{P}[\mathbf{u}] = \int_Q e^{i\mathbf{q}\cdot\mathrm{x}}dQ\left\{\frac{1}{\Delta t_m}\int_t^{t+\Delta t_m} \mathbf{P}[\mathbf{f}(\mathbf{q};\tau) - \mathbf{f}(\mathbf{q}; t)]d\tau\right\}. \tag{16}$$

in the sense of iterated Lebesgue integration. Since $\boldsymbol{P}$ is a matrix independent of t, while $\boldsymbol{f}(\boldsymbol{q};\,t)$ depends continuously on t, it is clear that, for each fixed $\boldsymbol{q}$,

$$\frac{1}{\Delta t_m}\int_t^{t+\Delta t_m} \boldsymbol{P}[\boldsymbol{f}(\boldsymbol{q};\,\tau) - \boldsymbol{f}(\boldsymbol{q};\,t)]d\tau = \boldsymbol{\Delta}_m(\boldsymbol{q}) \to 0 \text{ as } \Delta t_m \to 0.$$

Hence, by Lebesgue's Dominated Convergence Theorem and (16), $\Delta \boldsymbol{u}/\Delta t_m \to \boldsymbol{P}[\boldsymbol{u}]$ for each fixed $\mathbf{x}$ provided

$$(17) \qquad |\boldsymbol{\Delta}_m(\boldsymbol{q})| \leqq K(\boldsymbol{q}), \qquad \int_Q K(\boldsymbol{q})dQ < +\infty.$$

But, by definition,

$$|\Delta t_m \boldsymbol{\Delta}_m|^2 = \sum_{j=1}^n \left| \int_t^{t+\Delta t_m} \sum_{k=1}^n p_{jk}(i\boldsymbol{q})[f_k(\boldsymbol{q},\,\tau) - f_k(\boldsymbol{q};\,t)]d\tau \right|^2.$$

Applying Schwarz' Inequality, we get

$$\begin{aligned} |\Delta t_m \boldsymbol{\Delta}_m| &\leqq \sum_{j=1}^m \int_t^{t+\Delta t_m} (\sum |p_{jk}|^2)^{1/2}(|\boldsymbol{f}(\tau)| + |\boldsymbol{f}(t)|)d\tau \\ &\leqq 2M(\boldsymbol{q})\Delta t_m \quad \text{by (13*)}. \end{aligned}$$

This gives (17), with $K(\boldsymbol{q}) = 2M(\boldsymbol{q})$.

7. **Concrete Cauchy problem.** Suitably combining Theorem 3 with the methods used in proving Theorems 1–2, we can also solve the concrete Cauchy problem. First, we observe that, in §4, the results are unchanged if we replace $C(\boldsymbol{q})$ by $w(\boldsymbol{q})C(\boldsymbol{q})$ and $C(\boldsymbol{q}_0)$ by $w(\boldsymbol{q}_0)C(\boldsymbol{q}_0)$, for *any* positive scalars $w(\boldsymbol{q})$ and $w(\boldsymbol{q}_0)$. In particular, the modulus of the $T_t(\boldsymbol{q})$ is unchanged, while $N(\boldsymbol{q};\,\boldsymbol{\phi})$ is replaced by $N_w(\boldsymbol{q};\,\boldsymbol{\phi}) = w(\boldsymbol{q})N(\boldsymbol{q};\,\boldsymbol{\phi})$.

By choosing the $w(\boldsymbol{q})$ sufficiently large, however, we can make semi-orbits in the complex Banach space $B(N_w)$ of the $\boldsymbol{\phi}(\boldsymbol{q}) \in \Phi$ such that (cf. (8))

$$\int N(\boldsymbol{q};\,\boldsymbol{\phi}(\boldsymbol{q})w(\boldsymbol{q})dQ < +\infty,$$

consist entirely of functions which satisfy the conditions of §6.

More precisely, any Borel function $C(\boldsymbol{q})$ of the Corollary to Theorem 2 can be modified in the following way. For $\boldsymbol{q} \in Q$, $\boldsymbol{q}$ is in some Q_h and $C(\boldsymbol{q}) = C(\boldsymbol{q}_h)$ where $Q_h \subseteq R_h = R(\boldsymbol{q}_h)$. Let $w(\boldsymbol{q}) = w(\boldsymbol{q}_h)$ where $w(\boldsymbol{q}_h) > 0$ is so chosen that in $R(\boldsymbol{q}_h)$

$$(18) \qquad w(q_h)N(q_h;\, T_t(q)[\phi]) \geqq (1 + \sum |p_{jk}(iq)|)\cdot |\phi| .$$

Then $w(q)$ is a Borel function since it is constant on each Borel set Q_h, and $w(q)C(q)$ defines a Banach space $B(N_w)$ as before. The modulus of the semigroup (5) is given by

$$(19) \qquad \|T_t(q)\| N_w \leqq e^{\sigma t},$$

where

$$(20) \qquad N_w(q;\phi) = w(q)N(q;\phi).$$

It will follow that, on any interval $|t-\tau| < \epsilon$ of any semi-orbit in $B(N_w)$, the hypotheses of Theorem 3 will be satisfied, with $M(q) = \sup_\tau e^{\sigma\tau} N_w(q;\phi(q))$. This proves

THEOREM 4. *For suitable $B(N)$, every semi-orbit in the corollary of Theorem 2 represents an actual solution of* (1).

We now define the system (1) to be *hyperbolic* if and only if P and $-P$ are both regular (see [1, §9] for a discussion of alternative definitions). Since the substitution of $-P$ for P leaves the modified Jordan canonical form of §4 unchanged except for sign, (10) holds with the substitution of $|t|$ for t. This proves the

COROLLARY. *If* (1) *is hyperbolic, then the C_0-semigroups of Theorem* 4 *and Corollary of Theorem* 2 *are parts of C_0-groups, with moduli $\|T_t\| \leqq e^{\sigma|t|}$.*

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