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ON THE IDENTITY OF FUNCTION SPACES ON CARTESIAN PRODUCT SPACES

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For $i = 1, \ldots, n$, let $S_i$ be a compact Hausdorff space and $F_i$ be a closed linear subspace of the complex Banach space $C(S_i)$, the set of all continuous functions from $S_i$ to the complex numbers. Let $S_1 \times \cdots \times S_n$ be the Cartesian Product of $S_1, \ldots, S_n$.

Define $F_1 \star \cdots \star F_n$ as $\{ \phi | \phi \in C(S_1 \times \cdots \times S_n); \text{for any } i = 1, \ldots, n \text{ and } (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n), \phi(s_1, \ldots, s_{i-1}, \ldots, s_{i+1}, \ldots, s_n) \in F_i \}$. Also, define $F_1 \otimes \cdots \otimes F_n$ as the closure of the space of linear combinations of functions of the form $\phi(s_1, \ldots, s_n) = f_1(s_1) \times \cdots \times f_n(s_n)$ where each $f_i \in F_i$, where we base the topology on the norm, $\|\phi\| = \max_{s_1, \ldots, s_n} |\phi(s_1, \ldots, s_n)|$.

It is easily shown that $F_1 \otimes \cdots \otimes F_n$ is a subspace of $F_1 \star \cdots \star F_n$ and that if $F_2$ is one-dimensional, then $F_1 \otimes F_2 = F_1 \star F_2$. Furthermore, by using continuous partitions of unity, one may show that $F_1 \otimes C(S_2) = F_1 \star C(S_2)$. Therefore, if all but at most one of the $F_i$ are either one-dimensional or $C(S_i)$, then $F_1 \otimes \cdots \otimes F_n = F_1 \star \cdots \star F_n$. However, it is not known whether for all cases $F_1 \otimes \cdots \otimes F_n$ will equal $F_1 \star \cdots \star F_n$ or not.\footnote{Under Proposition 37 of his first paper of the Amer. Math. Soc. Memoirs, no. 16, Alexandre Grothendieck discusses a number of conjectures which are equivalent to this one.} Although this question is not fully answered here, the purpose of this paper is to give a partial answer to this question. The results and arguments of this paper also apply to real-valued function spaces.

1. **Lemma.** Let $F$ be a closed linear subspace of $C(S_1)$ and $G$ a closed linear subspace of $C(S_2)$. Let $H$ be a closed subspace of $G$ differing from $G$ by only one dimension. Then $F \otimes G = F \star G$ implies that $F \otimes H = F \star H$.\footnote{A small part of the work done under an AEC Predoctoral Fellowship at Yale University, year 1952–1953, under the kind and patient guidance of Dr. Charles E. Rickart.}
Proof. Choose a continuous linear functional \( x^* \) on \( G \) such that \( \| x^* \| = 1 \) and \( x^*(h) = 0 \) for \( h \in H \). Choose \( g_0 \in G \) such that \( x^*(g_0) = 1 \). Then every element of \( G \) will be of the form \( h + \lambda g_0 \) where \( h \in H \) and \( \lambda \) is a complex number.

Choose \( \phi \in F \ast H \) and \( \epsilon > 0 \). Since \( \phi \in F \ast G = F \otimes G \), choose \( f_i \in F \), \( g_i \in G \), \( i = 1, \ldots, k \) such that

\[
\| \phi - \sum_{i=1}^{k} f_i \times g_i \| < \epsilon (1 + \| g_0 \|)^{-1}.
\]

However, each \( g_i \) may be expressed as \( h_i + \lambda_i g_0 \) where \( h_i \in H \). Therefore,

\[
\sum_{i=1}^{k} f_i \times g_i = \sum_{i=1}^{k} f_i \times (h_i + \lambda_i g_0) = y + z \times g_0
\]

where \( y \in F \otimes H \) and \( z \in F \).

For each \( s \in S_1 \),

\[
| z(s) | = | x^*[z(s)g_0] | = | x^*[y(s, \cdot) + z(s)g_0 - \phi(s, \cdot)] | \\
\leq \| y + z \times g_0 - \phi \| \| x^* \| < \epsilon (1 + \| g_0 \|)^{-1}.
\]

Therefore,

\[
\left\| \sum_{i=1}^{k} f_i \times h_i - \phi \right\| = \| y - \phi \|
\]

\[
\leq \| y + z \times g_0 - \phi \| + \| z \times g_0 \| = \| y + z \times g_0 - \phi \| + \| z \| \| g_0 \|
\]

\[
< \epsilon (1 + \| g_0 \|)^{-1} + \epsilon \| g_0 \| (1 + \| g_0 \|)^{-1} = \epsilon.
\]

Since \( f_i \in F \) and \( h_i \in H \), it follows that \( \phi \in F \otimes H \). Therefore, \( F \otimes H = F \ast H \).

2. Lemma. Let \( S_1, S_2, F, G \) and \( H \) be as in Lemma 1. Then \( F \otimes H = F \ast H \) implies that \( F \otimes G = F \ast G \).

Proof. Choose \( g_0 \in G - H \) and \( x^* \) as in the proof of Lemma 1. Since the set of continuous linear functionals on \( C(S_2) \) is the set of regular measures on \( S_2 \), we may choose a measure \( \mu \) such that \( \int_{S_2} g(s_2) \mu(ds_2) = x^*(g) \) for each \( g \in G \).

Consider an arbitrary element \( \phi \in F \ast G \). For \( s_1 \in S_1 \), define \( \psi(s_1) \) as

\[
\int_{S_2} \phi(s_1, s_2) \mu(ds_2).
\]
Let $\mu'$ be any regular measure on $S_1$ such that for $f \in F$, $\int_{S_1} f(s_1) \mu'(ds_1) = 0$. Then using Fubini's Theorem, we get that
\[
\int_{S_1} \psi(s_1) \mu'(ds_1) = \int_{S_1} \int_{S_2} \phi(s_1, s_2) \mu(ds_2) \mu'(ds_1) = \int_{S_2} \int_{S_1} \phi(s_1, s_2) \mu'(ds_1) \mu(ds_2) = \int_{S_2} 0 \mu(ds_2) = 0.
\]
Therefore, $\psi \in F$.

For any $s_1 \subseteq S_1$, it follows that
\[
\int_{S_2} [\phi(s_1, s_2) - \psi(s_1)g_0(s_2)] \mu(ds_2) = \psi(s_1) - \psi(s_1) = 0
\]
and so $\phi(s_1, \cdot) - \psi(s_1)g_0(\cdot)$ is an element of $H$. Therefore $\phi - \psi \times g_0 \in F \star H = F \otimes H \subseteq F \otimes G$. Consequently, $\phi = [\phi - \psi \times g_0] + \psi \times g_0$ is an element of $F \otimes G$.

3. Theorem. For each $i = 1, \cdots, n$, let $G_i$ and $F_i$ be closed linear subspaces of $C(S_i)$ which differ from each other by at most a finite number of dimensions. Let no $F_i$ or $G_i$ be zero-dimensional.

Then $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$ if and only if $G_1 \otimes \cdots \otimes G_n = G_1 * \cdots * G_n$.

Proof. Let $H_1, \cdots, H_k$ be closed linear subspaces of $C(S_i)$ such that $H_1 = F_1$, $H_k = G_1$ and for $i = 1, \cdots, k - 1$, $H_i$ and $H_{i+1}$ differ from each other by only one dimension. If $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$, then $F_2 \otimes \cdots \otimes F_n = F_2 * \cdots * F_n$. Using Lemma 1 or 2, we see that $H_2 \otimes F_2 \otimes \cdots \otimes F_n = H_2 * F_2 * \cdots * F_n$. Applying the lemmas in this way, a total of $k - 1$ times, we see that $G_1 \otimes F_2 \otimes \cdots \otimes F_n = G_1 * F_2 * \cdots * F_n$. Now repeating this procedure $n - 1$ more times, we obtain that $G_1 \otimes \cdots \otimes G_n = G_1 * \cdots * G_n$.

By repeating the above arguments, we see that $G_1 \otimes \cdots \otimes G_n = G_1 * \cdots * G_n$ implies $F_1 \otimes \cdots \otimes F_n = F_1 * \cdots * F_n$ and hence the theorem is proved.

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