

ON NONCOMMUTATIVE JORDAN ALGEBRAS

R. D. SCHAFER¹

1. Introduction. Let A be a power-associative algebra of finite dimension over a field F , and let A have a unity element 1 . Suppose that every element of A has the form $\alpha 1 + z$ where $\alpha \in F$ and z is nilpotent. For some classes of algebras—for example, commutative Jordan algebras of characteristic $\neq 2$, alternative (including associative) algebras of arbitrary characteristic, noncommutative Jordan algebras of characteristic 0 —it follows then that $A = F1 + N$ where N is a nil subalgebra (ideal) of A .

However, this is not always the case, as may be seen from an example. Let M be a vector space over F on which is defined a nonzero alternate bilinear form ϕ . Let $A = F1 + M$ where multiplication is defined by $xy = \phi(x, y)1$ for x, y in M . Then every element a of A may be written as $a = \alpha 1 + z$ where $z^2 = 0$. Also A , being a quadratic algebra, is power-associative. If N is a nil subalgebra of A , then $N \subseteq M$, and $A = F1 + N = F1 + M$ implies $N = M$. However, M is not a subalgebra of A since there exist x, y in M such that $xy = \phi(x, y)1 \notin M$.

We shall call a power-associative algebra A with 1 over F a *nodal algebra* in case every element of A is of the form $\alpha 1 + z$ where $\alpha \in F$ and z is nilpotent, and A is not of the form $A = F1 + N$ for N a nil subalgebra of A .

Jacobson has proved [4]² the nonexistence of nodal commutative Jordan algebras (of characteristic $\neq 2$). This has closed the one remaining gap in Albert's structure theory for commutative Jordan algebras by eliminating the possibility of new simple commutative Jordan algebras of characteristic p (which would necessarily have been nodal algebras).

We have given in [5] a structure theory for noncommutative Jordan algebras of characteristic 0 . In this paper we give a structure theory for those noncommutative Jordan algebras of characteristic $\neq 2$ for which A_K is without nodal subalgebras (where K is the algebraic closure of F). We conclude with some theorems about nodal noncommutative Jordan algebras. L. A. Kokoris has informed us, just as this paper is being submitted for publication, that he has succeeded in constructing examples of simple noncommutative Jordan algebras of characteristic p which are nodal algebras.

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² Numbers in brackets refer to the references cited at the end of the paper.

We use throughout the definitions and notations of our earlier paper [5]³ on noncommutative Jordan algebras of characteristic 0, except that we need a more refined definition of trace-admissibility as given by Albert in [3]. We assume throughout that F is of characteristic $\neq 2$.

2. Trace-admissible algebras. Let A be a power-associative algebra over F , and K be the algebraic closure of F . A linear function $\delta(x)$ on A_K to K is called an *admissible trace function* for A , and A is called *trace admissible*, in case:

- (I) $\delta(xy) = \delta(yx)$ for all x, y in A ;
- (II) $\delta((xy)z) = \delta(x(yz))$ for all x, y, z in A ;
- (III) $\delta(u) \neq 0$ for every primitive idempotent u in A_K ;
- (IV) $\delta(z) = 0$ for every nilpotent element in A_K .

In [3] a *primitive trace function* for any special Jordan algebra was constructed. Here we require the generalization to any commutative Jordan algebra.

Let B be a commutative Jordan algebra over F , M the radical of B_K , and $S = B_K/M = S_1 \oplus \cdots \oplus S_r$, where the S_i are simple ideals of S . For x in B , we have the residue class $\bar{x} = \bar{x}_1 + \cdots + \bar{x}_r$ with \bar{x}_i in S_i . If S_i is special, define $\delta_i(x_i)$ as in [3, p. 410]. Otherwise S_i is the (27-dimensional) algebra (under "quasimultiplication") of all 3-rowed hermitian matrices

$$X = \begin{pmatrix} \xi_1 & c_3 & \bar{c}_2 \\ \bar{c}_3 & \xi_2 & c_1 \\ c_2 & \bar{c}_1 & \xi_3 \end{pmatrix}$$

with elements in the Cayley algebra over K , and we define $\delta_i(X) = \xi_1 + \xi_2 + \xi_3$ in K . Then $\delta_i(X)$ is an admissible trace function for S_i . For (I) is obvious and the well-known formula (II) may be computed in a straightforward way. (III) and (IV) may be seen from the equation

$$X^3 - \delta_i(X)X^2 + \beta(X)X + \gamma(X)1 = 0$$

where $2\beta(X) = [\delta_i(X)]^2 - \delta_i(X^2)$ and $6\gamma(X) = 3\delta_i(X)\delta_i(X^2) - [\delta_i(X)]^3 - 2\delta_i(X^3)$ [6, p. 763]. Since $F[X]$ is an associative algebra, the minimal polynomial of X divides $f(\lambda) = \lambda^3 - \delta_i(X)\lambda^2 + \beta(X)\lambda + \gamma(X)$. Hence, if $X \neq 1$ is idempotent, we have⁴ $\delta_i(X) = 1$ or 2 while, if X is

³ We take this opportunity to correct a misprint on p. 473 of [5]. The final commutator in line 19 should be $[2Lx^2 - L_x^2 + R_x^2, R_x]$.

⁴ Actually $\delta_i(u) = 1$ for any primitive idempotent u in case the characteristic is $\neq 3$, for then $1 = u + e + f$ for idempotents e and f forces $\delta_i(u) = \delta_i(e) = \delta_i(f) = 1$ since $\delta_i(1) = 3$. However, if one were content to assume characteristic $\neq 3$, the trace-admissibility of $\delta_i(X)$ would follow directly from that of trace $R_X = 9\delta_i(X)$.

nilpotent, it follows that $\delta_i(X) = 0$. For x in B , define

$$\delta(x) = \delta_1(\bar{x}_1) + \cdots + \delta_r(\bar{x}_r),$$

the primitive trace function of B .

THEOREM 1. *The primitive trace function of any commutative Jordan algebra B over F is an admissible trace function for B .*

For (I) is obvious, and (II) follows directly from the direct sum relationship in $S = S_1 \oplus \cdots \oplus S_r$. If $u \in B_K$ is a primitive idempotent, we use [4] to observe that Albert's proof in [3, p. 411] that \bar{u} is a primitive idempotent in one of the S_i is valid. If $z \in B_K$ is nilpotent, then $\bar{z} = \bar{z}_1 + \cdots + \bar{z}_r$ for nilpotent \bar{z}_i , and $\delta(z) = 0$.

THEOREM 2. *Let A be a noncommutative Jordan algebra over F . Assume that A_K (K the algebraic closure of F) is without nodal subalgebras. Then the primitive trace function $\delta(x)$ of $B = A^+$ is an admissible trace function for A .*

We may assume that $1 \in A$. Since powers in A and A^+ coincide, (III) and (IV) are immediate. Assuming that (I) has been proved, (II) follows directly from the flexible law [5, p. 474]. Now (I) holds in A if and only if it holds in A_K , so we may assume that F is algebraically closed and that A itself contains no nodal subalgebra.

Let $1 = e_1 + \cdots + e_t$ for pairwise orthogonal primitive idempotents in A . Then A is the direct sum $A = \sum_{i \leq j} A_{ij}$ where

$$\begin{aligned} A_{ii} &= \{a \mid e_i a = a e_i = a\} = \{a \mid e_i \cdot a = a\}, \\ A_{ij} &= \{a \mid e_i \cdot a = a \cdot e_j = a/2\} \end{aligned} \quad \text{for } i \neq j.$$

Using the notation $a_{\alpha\beta} \in A_{\alpha\beta}$, we see that it is sufficient to prove that $\delta(x_{ij}y_{hk}) = \delta(y_{hk}x_{ij})$. We are concerned with six cases:

- (i) i, j, h, k distinct;
- (ii) $i = j, h \neq i, k \neq i$;
- (iii) $h = j; i, j, k$ distinct;
- (iv) $h = j = k \neq i$;
- (v) $h = i \neq j = k$;
- (vi) $h = i = j = k$.

In cases (i) and (ii) it is known [1, p. 560] that $A_{ij}A_{hk} = A_{hk}A_{ij} = 0$. If u is any idempotent in A , application of [3, Lemma 1] to A^+ yields $\delta(a) = 0$ for every $a \in A_u(1/2) = \{a \mid a \cdot u = a/2\}$. In case (iii) we let $u = e_i + e_j$. Then $x_{ij} \in A_u(1) = \{a \mid au = ua = a\}$, $y_{jk} \in A_u(1/2)$. But $A_u(1)A_u(1/2) \subseteq A_u(1/2)$ and $A_u(1/2)A_u(1) \subseteq A_u(1/2)$ since A is stable [1, p. 562] so that $\delta(x_{ij}y_{jk}) = \delta(y_{jk}x_{ij}) = 0$. In case (iv) we have similarly $\delta(x_{ij}y_{jj}) = \delta(y_{jj}x_{ij}) = 0$. In case (v) let $x = x_{ij}$ and $y = y_{ij}$ in A_{ij} .

Then $x \cdot y \in A_{ii} + A_{jj}$ implies

$$xy = w_{ii} + w_{ij} + w_{jj}, \quad yx = z_{ii} - w_{ij} + z_{jj}.$$

Let $e_i y = t \in A_{ij}$ so that $y e_i = y - t$. Then flexibility implies that $0 = (x, y, e_i) + (e_i, y, x) = w_{ii} + w_{ij} e_i - xy + xt + tx - z_{ii} + e_i w_{ij} = 2x \cdot t - z_{ii} - w_{jj}$, so that

$$(1) \quad 2x \cdot t = z_{ii} + w_{jj}.$$

Also

$$(2) \quad 2x \cdot y = (w_{ii} + z_{ii}) + (w_{jj} + z_{jj}).$$

Applying [3, Lemma 1] to (1) and (2), we have $\delta(z_{ii}) = \delta(w_{jj})$ and $\delta(w_{ii}) + \delta(z_{ii}) = \delta(w_{jj}) + \delta(z_{jj})$ so that $\delta(w_{ii}) = \delta(z_{jj})$. Hence $\delta(xy) = \delta(w_{ii}) + \delta(w_{jj}) = \delta(z_{jj}) + \delta(z_{ii}) = \delta(yx)$, completing the proof in case (v). It is only for case (vi) that we require that A be without nodal subalgebras. Since the unity element e_i of the subalgebra A_{ii} is an absolutely primitive idempotent, the commutative Jordan algebra $A_{ii}^+ = e_i F + N^+$ where N^+ is a nil subalgebra of A_{ii}^+ . Then $A_{ii} = e_i F + N$, and every element of A_{ii} has the form $x = \alpha e_i + z$ where $\alpha \in F$ and z is nilpotent. Also N consists of the nilpotent elements of A_{ii} . Since A_{ii} is not a nodal algebra, N is a subalgebra of A_{ii} . If $y = \beta e_i + w$ for $w \in N$, then zw and wz are nilpotent, and $\delta(xy) = \alpha\beta\delta(e_i) = \delta(yx)$.

THEOREM 3. *Let A be as in Theorem 2. Then the radical N of A coincides with the radical of A^+ and consists of those elements z satisfying $\delta(xz) = 0$ for every x in A . Also A/N is semisimple, and $(A/N)_K = A_K/N_K$ is without nodal subalgebras.*

The first conclusion follows from [3, Theorems 1, 2]. Also any nil ideal of A/N is clearly 0. Suppose that A_K/N_K contains a nodal subalgebra C with unity element u . Then $C = Ku + M$ with M^+ a nil subalgebra of C^+ , but there exist $m_i \in M$ such that $m_1 m_2 = \mu u + m_3$ with $\mu \neq 0$. Let D be the complete inverse image of C in A_K (under the natural homomorphism of A_K onto A_K/N_K). By the power-associativity of D there is an idempotent e in D such that $\bar{e} = u$. Clearly e is (absolutely) primitive in D . Then $D_e(1)$ is a nodal subalgebra of A_K , a contradiction. For $D_e(1) = Ke + R$ where R^+ is a nil subalgebra of $D_e(1)^+$. It follows that the subspace R consists of all nilpotent elements of $D_e(1)$ and it is sufficient to show that R is not a subalgebra of $D_e(1)$. There exist $a_i \in D_e(1)$ such that $\bar{a}_i = m_i$. For there are $d_i \in D$ with $\bar{d}_i = m_i$, and $D = D_e(1) + D_e(1/2) + D_e(0)$ implies $d_i = a_i + b_i + c_i$, $a_i \in D_e(1)$, $b_i e + e b_i = b_i$, $e_i c = 0$. Hence $\bar{b}_i u + u \bar{b}_i = \bar{b}_i = 2\bar{b}_i$, $\bar{c}_i u = 0 = \bar{c}_i$, or $\bar{b}_i = \bar{c}_i = 0$, $\bar{a}_i = \bar{d}_i = m_i$. Actually $a_i \in R$ since the

m_i are nilpotent and N_K is nil. Hence $\bar{a}_1\bar{a}_2 = m_1m_2 = \mu u + m_3 = \mu\bar{e} + \bar{a}_3$, or $a_1a_2 \equiv \mu e + a_3 \pmod{N_K}$. But $a_1a_2 = \lambda e + r$, $r \in R$, so $(\lambda - \mu)e$ is nilpotent, $\lambda = \mu \neq 0$, $a_1a_2 \notin R$.

THEOREM 4. *Let A be as in Theorem 2. If A is semisimple, then A is uniquely expressible as a direct sum $A = A_1 \oplus \cdots \oplus A_r$ of simple ideals A_i . If A is simple, then A is one of the following: a simple (commutative) Jordan algebra, a simple flexible algebra of degree two, or a simple quasiassociative algebra.*

A^+ is semisimple. Hence A is a direct sum $A = A_1 + \cdots + A_r$ of subspaces A_i such that the A_i^\pm are simple ideals of A^+ . Let e_i be the unity element of A_i^\pm so $A_i = A_{e_i}(1)$. For any $x_i \in A_i$, $y \in A$, we have $(e_i, x_i, y) + (y, x_i, e_i) = 0$, or $x_iy - e_i(x_iy) + (yx_i)e_i - yx_i = 0$, so that $2x_iy - 2e_i \cdot (x_iy) = 2x_i \cdot y - 2(x_i \cdot y)e_i = 0$ since $x_i \cdot y \in A_i^\pm$. Hence $x_iy \in A_i$. Also $yx_i = 2x_i \cdot y - x_iy \in A_i$, so A_i is an ideal of A . Since any ideal of A_i is an ideal of A_i^\pm , it follows that A_i is simple.

If A is simple (and A_K is without nodal subalgebras), then A is J -simple (that is, A^+ is simple). For otherwise $r > 1$ above. Albert's classification of flexible J -simple algebras, given in [1, pp. 588–593] for F of characteristic 0, is valid for characteristic $\neq 2$. For characteristic 0 was used only for a determination of A^+ , and it has subsequently been shown [2; 4] that there are no new simple commutative Jordan algebras A^+ (of characteristic p).

3. Nodal algebras. Throughout this section A is taken to be a noncommutative Jordan algebra with 1 over F such that every element is of the form $\alpha 1 + z$ where $\alpha \in F$ and z is nilpotent. Whenever A is assumed in addition to be a nodal algebra, this is explicitly stated.

By [4] $A^+ = F1 + N^+$ where N^+ is a nil subalgebra of A^+ . Hence $A = F1 + N$, where N is a subspace consisting of all the nilpotent elements of A , and $x \cdot y \in N$ for all x, y in N . Then A is a nodal algebra if and only if there exist x, y in N such that $xy \notin N$. The subalgebra generated by two such elements is itself a nodal algebra. Clearly a nodal algebra cannot be trace-admissible. For, if $xy = \lambda 1 + z$ with $\lambda \neq 0 \in F$ and $z \in N$, then $0 = \delta(x \cdot y) = \delta(xy) = \lambda \delta(1) \neq 0$. (It follows from [5] that nodal algebras are of characteristic p . A separate proof of this fact is given below.)

Since N^+ is a nilpotent commutative Jordan algebra, the powers of N^+ lead to 0. If G and H are subspaces of A , we denote by $G \cdot H$ the subspace of A spanned by all $g \cdot h$ for $g \in G, h \in H$. Then, defining N_i by $N_1 = N, N_i = N_{i-1} \cdot N$, we have

$$(3) \quad N_1 \supset N_2 \supset N_3 \supset \cdots \supset N_k \supset N_{k+1} = 0,$$

where the inclusions are all proper since $N_i = N_{i+1}$ implies $N_i = N_{i+1} = N_{i+2} = \dots$. Also let $N^{\cdot i}$ denote the subspace of A^+ spanned by all products (that is, \cdot products) of i factors from N , no matter how associated. That is, $N^{\cdot 1} = N$, $N^{\cdot i} = \sum_{j=1}^{i/2} N_{i-j} \cdot N_j$. Then

$$N^{\cdot 1} \supset N^{\cdot 2} \supset N^{\cdot 3} \supseteq N^{\cdot 4} \supseteq \dots \supseteq N^{\cdot s} \supset N^{\cdot s+1} = 0,$$

where this series also terminates in 0, but there is no reason to suppose that the inclusions are all proper. We have $N_i \subseteq N^{\cdot i}$ and $N^{\cdot 2^{i-1}} \subseteq N_i$, so that $k \leq s \leq 2^k - 1$.

Let x, y be in N . Then

$$(4) \quad xy = \lambda 1 + z, \quad \lambda \in F, z \in N.$$

Hence

$$(5) \quad yx = -\lambda 1 + (2x \cdot y - z)$$

and $(xy)x = \lambda x + zx = x(yx) = -\lambda x + 2x(x \cdot y) - xz$, or

$$(6) \quad \lambda x = x(x \cdot y) - x \cdot z.$$

It follows that we have $N_2 \neq 0$ in any nodal algebra A , for $N_2 = 0$ implies $xy \in N$ for all $x, y \in N$ by (6).

Now $0 = (x, x, y) + (y, x, x) = x^2y - x(\lambda 1 + z) + (-\lambda 1 + 2x \cdot y - z)x - yx^2 = 2x^2y - 2\lambda x - 2x \cdot z + 4(x \cdot y) \cdot x - 2x(x \cdot y) - 2x^2 \cdot y$ implies

$$(7) \quad x^2y = 2\lambda x + 2x \cdot z - 2(x \cdot y) \cdot x + x^2 \cdot y$$

by (6). Linearization of (7), using $x_i y = \lambda_i 1 + z_i$, gives

$$(8) \quad (x_1 \cdot x_2) y = \lambda_1 x_2 + \lambda_2 x_1 + x_1 \cdot z_2 + x_2 \cdot z_1 - (x_1 \cdot y) \cdot x_2 - (x_2 \cdot y) \cdot x_1 + (x_1 \cdot x_2) \cdot y.$$

THEOREM 5.

- (a) $N_i N \subseteq N_{i-1}$, $NN_i \subseteq N_{i-1}$ for $i = 2, 3, \dots$;
- (b) $N_i N_2 \subseteq N_i$, $N_2 N_i \subseteq N_i$ for $i = 1, 2, \dots$;
- (c) $N_k N_2 = N_2 N_k = 0$ for k in (3).

COROLLARY. N_i is a nil subalgebra of A for $i = 2, 3, \dots$.

$N_2 N \subseteq N$ is implied by (8). Also $NN_2 \subseteq N \cdot N_2 + N_2 N \subseteq N$. For $i \geq 3$ in (a), we assume $N_{i-1} N \subseteq N_{i-2}$. Then $x_1 \in N_{i-1} \subseteq N_2$ implies $\lambda_1 = 0$, $z_1 = x_1 y \in N_{i-2}$ in (8) so that $N_i N = (N_{i-1} \cdot N) N \subseteq N_{i-1}$. Then $NN_i \subseteq N \cdot N_i + N_i N \subseteq N_{i-1}$. The case $i = 1$ of (b) has been proved in (a). For $i \geq 2$ we assume $N_{i-1} N_2 \subseteq N_{i-1}$. Then $y \in N_2$ implies $\lambda_1 = \lambda_2 = 0$ in (8) while $x_1 \in N_{i-1}$ implies $z_1 = x_1 y \in N_{i-1} N_2 \subseteq N_{i-1}$. Hence $N_i N_2 \subseteq N_i$ by (8), and $N_2 N_i \subseteq N_2 \cdot N_i + N_i N_2 \subseteq N_i$. In case $i = k$ as in (3) we may sharpen (b) to (c) as follows. We need only consider $k \geq 2$. Taking $y \in N_k$ in (7), we have $\lambda = 0, z = xy$, so that $x^2y = 2x \cdot (xy) = 2x(x \cdot y) = 0$

for all $x \in N$. Linearization of $x^2y=0$ gives $N_2N_k=0$, and $N_kN_2 \subseteq N_k \cdot N_2 + N_2N_k=0$. The corollary follows from (b) since $N_iN_i \subseteq N_iN_2 \subseteq N_i$ for $i \geq 2$.

We omit the proofs of similar relationships involving the spaces $N^{\cdot i}$:

$$\begin{aligned}
 N^{\cdot i}N &\subseteq N^{\cdot i-1} & NN^{\cdot i} &\subseteq N^{\cdot i-1} && \text{for } i = 2, 3, \dots; \\
 N^{\cdot h}N^{\cdot i} &\subseteq N^{\cdot h+i-2} &&&& \text{for } h + i = 3, 4, \dots
 \end{aligned}$$

These may be proved, using (8), by induction on i .

Equation (4) states that $xy \equiv \lambda 1 (= \lambda x^0) \pmod{N_1}$. We generalize this to

$$(9) \quad x^i y \equiv i \lambda x^{i-1} \pmod{N_i} \quad \text{for } i = 1, 2, \dots$$

in a proof by induction on i . It is known [1, p. 574] that the right and left multiplications of a and a^2 in A generate a commutative associative algebra which contains the right and left multiplications of all powers of a ; it also contains R_a^+ . Hence $(x^i, x, y) + (y, x, x^i) = 0$ implies $x^{i+1}y - \lambda x^i - x^i z - \lambda x^i + 2(x \cdot y)x^i - zx^i - yx^{i+1} = 0$, or $2x^{i+1}y = 2\lambda x^i + 2x^i \cdot z - 2x \cdot (yx^i) + 2x^{i+1} \cdot y \equiv 2\lambda x^i - 2x \cdot (-i\lambda x^{i-1}) \pmod{N_{i+1}}$ by the assumption of the induction. Hence $x^{i+1}y \equiv (i+1)\lambda x^i \pmod{N_{i+1}}$. Now (9) is the case $j=1$ of

$$(10) \quad x^i R_y^j \equiv i(i-1) \dots (i-j+1) \lambda^j x^{i-j} \pmod{N_{i-j+1}} \quad (1 \leq j \leq i).$$

We assume (10) and see that $x^i R_y^{j+1} \equiv i(i-1) \dots (i-j+1) \lambda^j x^{i-j} R_y \equiv i(i-1) \dots (i-j+1)(i-j) \lambda^{j+1} x^{i-j-1} \pmod{N_{i-j}}$ by (9) and Theorem 5(a). In particular, we have

$$(11) \quad x^i R_y^{i-1} \equiv i! \lambda^{i-1} x \pmod{N_2}.$$

We also generalize (6) to

$$(12) \quad \lambda x^i = y(R_x^+)^i L_x - z(R_x^+)^i \quad \text{for } i = 1, 2, 3, \dots$$

by induction on i . Assuming (12), we have

$$\lambda x^{i+1} = \lambda x^i \cdot x = y(R_x^+)^i L_x R_x^+ - z(R_x^+)^{i+1} = y(R_x^+)^{i+1} L_x - z(R_x^+)^{i+1}.$$

If A is a nodal algebra and \bar{A} is any nonzero homomorphic image of A , then \bar{A} is also a nodal algebra. For any ideal $T \neq A$ in $A = F1 + N$ is contained in N . Otherwise there exists $t = \tau 1 + z$ in T with $\tau \neq 0$ in F and z in N . Power-associativity implies that t^{-1} exists in A , so that $tt^{-1} = 1 \in T$, or $T = A$, a contradiction. Then $\bar{A} \cong A/T = F\bar{1} + N/T$ where N/T consists of all of the nilpotent elements of A/T , and

$\bar{A} \cong A/T$ is a nodal algebra since there exist x, y in N with $xy = \lambda 1 + z$, $\lambda \neq 0$, or $\bar{x}\bar{y} = \lambda \bar{1} + \bar{z} \in N/T$ although $\bar{x}, \bar{y} \in N/T$. It follows that, for any nodal algebra A over F , there is a simple nodal algebra over F (which is a homomorphic image of A).

THEOREM 6. *Let A be a nodal simple noncommutative Jordan algebra over F . Then F is of characteristic p , and p divides the dimension of A .*

It has been shown [5, p. 474], using only characteristic $\neq 2$, that $\delta(a) = \text{trace } R_a^+$ satisfies conditions (I), (II) for trace-admissibility. Hence, if $T = \{t \mid \delta(at) = 0 \text{ for all } a \in A\}$, T is easily seen to be an ideal of A . Hence $T = 0$ or $T = A$. But $T \supseteq N_2 \neq 0$ since $y \in N_2, a \in A$ imply $x = ay \in N$ by Theorem 5(a), or $\delta(ay) = \text{trace } R_{ay}^+ = \text{trace } R_x^+ = 0$ so that $y \in T$. Hence $T = A$, $\delta(ab) = 0$ for all $a, b \in A$. In particular, $\delta(1) = \text{trace } R_1^+ = \text{trace } I = 0$ where I is the identity transformation on A . The conclusion follows.

THEOREM 7. *If A is a nodal noncommutative Jordan algebra over F of characteristic p , then $N_p \neq 0$ (that is, $p \leq k$ in (3)).*

Since A is nodal, there exist x, y in N such that $\lambda \neq 0$ in (4). Take $i = k$ in (12) and (11). Since $y(R_x^+)^k = z(R_x^+)^k = 0$ by (3), we have $\lambda x^k = 0$, or $x^k = 0$. But then $k! \lambda^{k-1} x \in N_2$. If $p > k$, we have x in N_2 , xy in N by Theorem 5(a), a contradiction. Hence $p \leq k$, or $N_p \neq 0$.

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