RESTRICTED NONCOMMUTATIVE JORDAN ALGEBRAS OF CHARACTERISTIC p

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Noncommutative Jordan algebras over a field F are defined by the identities

$$(xa)x = x(ax), \tag{1}$$

$$(x^2a)x = x^2(ax). \tag{2}$$

In [7][2] we obtained a satisfactory structure theory for these algebras of finite dimension over F of characteristic 0 by proving that they are trace-admissible. Recent examples by L. A. Kokoris [**4**] show that the algebras satisfying (1) and (2) are not in general trace-admissible if F is of characteristic $p>2$.

It is natural to seek a generalization of (commutative) Jordan algebras of characteristic $p>2$ in which the algebras are trace-admissible. We find this generalization in the algebras satisfying (1), (2), and

$$R_x^p = R_{x^p} \tag{3}$$

where R_x denotes the right multiplication determined by x. We shall call an algebra over F of characteristic p, in which (1), (2), (3) are satisfied, a *restricted noncommutative Jordan algebra* (*of characteristic p*), using this terminology because of the analogy with restricted Lie algebras [**2**].

Commutative Jordan algebras of characteristic $p>2$ satisfy (3). For in any special Jordan algebra, the right multiplication is $R_x^+ = 1/2(R_x+L_x)$ where R_x and L_x are restrictions of right and left multiplications in an associative algebra. Then $(R_x^+)^p = 1/2^p(R_x+L_x)^p = 1/2(R_x^p+L_x^p)$ since $[R_x, L_x]=0$. Hence $(R_x^+)^p = 1/2(R_{x^p}+L_{x^p}) = R_{x^p}^+$ since powers in the special Jordan algebra and the associative algebra coincide. But (3) is an identity, $(\cdots((ax)x)\cdots)x = ax^p$, in only two elements. Since the free (commutative) Jordan algebra of characteristic $\neq 2$ with two generators is special [**9**], identity (3) is satisfied in any (commutative) Jordan algebra of characteristic $p>2$. (Or one may use the formal identity

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$$(4) \qquad R_{a^n} = 1/2\{(R_a + (R_{a^2} - R_a^2)^{1/2})^n + (R_a - (R_{a^2} - R_a^2)^{1/2})^n\}$$

in commutative Jordan algebras [**5**, p. 256] and the fact that $[R_a, R_{a^2}] = 0$ to see that $R_{a^p} = 1/2\{(R_a+C)^p + (R_a-C)^p\} = 1/2(R_a^p + C^p + R_a^p - C^p) = R_a^p$.)

Alternative algebras (of arbitrary characteristic) satisfy (3), as well as (1) and (2). In fact, if $p=2$, then (1) and (3) imply the alternative law (so that (2) holds), and the theory of these algebras is well known. Our proofs are given for $p>2$, but the statements of the results include $p=2$ when applicable.

LEMMA 1. *Let A be a restricted noncommutative Jordan algebra of characteristic p. If x is nilpotent, then R_x and L_x are nilpotent.*

For there is an integer e such that $x^{p^e}=0$. Hence (3) implies that $R_x^{p^e}=0$, R_x is nilpotent. Since A^+ is a commutative Jordan algebra, we know that $(R_x^+)^{p^f} = 0$ for some $f \geqq e$, or $1/2^{p^f}(R_x + L_x)^{p^f} = 1/2(R_x^{p^f} + L_x^{p^f}) = 1/2 L_x^{p^f} = 0$ since $[R_x, L_x]=0$ by (1).

A noncommutative Jordan algebra A is a *nodal algebra* [**8**] in case $A = F1 + N$ where N is a subspace consisting of the nilpotent elements of A, and $x \cdot y = 1/2(xy+yx) \in N$ for all $x, y \in N$, but there exist $x, y \in N$ such that $xy \notin N$. In [**4**] Kokoris constructs simple nodal noncommutative Jordan algebras for each characteristic $p>2$. We use here the notation of [**8**, §3]. In particular, we write the descending chain of subspaces

$$(5) \qquad N_1 \supset N_2 \supset N_3 \supset \cdots \supset N_k \supset N_{k+1} = 0$$

as in [**8**, (3)].

LEMMA 2. *Let $A = F1 + N$ be a noncommutative Jordan algebra of characteristic $\neq 2$. If $z \in N$, $w \in N_2$, and R_w is nilpotent, then $R_z^+ + R_w$ is nilpotent.*

We prove by induction on h that, for any i,

$$(6) \qquad N_i(R_z^+ + R_w)^h \subseteq N_{i+1} + N_i R_w^h.$$

The case $h=0$ is clear. Assuming (6), we have $N_i(R_z^+ + R_w)^{h+1} \subseteq (N_{i+1} + N_i R_w^h)(R_z^+ + R_w) \subseteq N_{i+2} + (N_i R_w^h)\cdot N + N_{i+1}N_2 + N_i R_w^{h+1}$. But $N_i R_w^h \subseteq N_i$ and $N_{i+1}N_2 \subseteq N_{i+1}$ by [**8**, Theorem 5(b)]. Hence $N_i(R_z^+ + R_w)^{h+1} \subseteq N_{i+1} + N_i R_w^{h+1}$. Let $R_w^t = 0$. Then (6) implies

$$(7) \qquad N_i(R_z^+ + R_w)^t \subseteq N_{i+1}$$

for any i. Hence

$$N_1(R_z^+ + R_w)^{kt} \subseteq N_2(R_z^+ + R_w)^{(k-1)t} \subseteq \cdots \subseteq N_k(R_z^+ + R_w)^t \subseteq N_{k+1} = 0$$

by (5), and $A(R_z^+ + R_w)^{kt+1} = (F1 + N_1)(R_z^+ + R_w)^{kt+1} \subseteq N_1(R_z^+ + R_w)^{kt} = 0$.

LEMMA 3. *Let $A = F1 + N$ be a noncommutative Jordan algebra of characteristic $\neq 2$. If $x, y \in N$ are such that $R_{x \cdot y}$ is nilpotent, then $xy \in N$.*

Let $y^r \neq 0$, $y^{r+1} = 0$. Then $xy = \lambda 1 + z$, $\lambda \in F$, $z \in N$, and $yx = -\lambda 1 + (2x \cdot y - z)$. Flexibility implies $(x, y, y^r) + (y^r, y, x) = 0$, so that $\lambda y^r + zy^r - xy^{r+1} + y^{r+1}x + \lambda y^r - 2y^r(x \cdot y) + y^r z = 0$, or $y^r(\lambda I + R_z^+ - R_{x \cdot y}) = 0$. Hence $\lambda I + R_z^+ - R_{x \cdot y}$ is singular. Let $w = -x \cdot y$ in Lemma 2, so that $R_z^+ - R_{x \cdot y}$ is nilpotent. This forces $\lambda = 0$, $xy \in N$.

Lemmas 1 and 3 imply

THEOREM 1. *A restricted noncommutative Jordan algebra of characteristic p cannot be a nodal algebra.*

If A is a restricted noncommutative Jordan algebra of characteristic p, then so is A_K where K is any extension of F. For there are at least p distinct elements in F, sufficiently many to effect the linearization of (3). Using the linearized form of (3), together with the intermediate identities obtained in the linearization, one can verify (3) in A_K. Then [**8**, Theorems 2, 3] imply

THEOREM 2. *Any restricted noncommutative Jordan algebra of characteristic p is trace-admissible. If $p > 2$, then the primitive trace function $\delta(x)$ of A^+ is an admissible trace function for A, and the radical N of A consists of those elements z satisfying $\delta(xz) = 0$ for every x in A.*

R. H. Oehmke [6] has recently shown that any semisimple noncommutative Jordan algebra of characteristic $\neq 2, 3$ is a direct sum of simple algebras, each of which is either a nodal algebra over its center or one of the simple algebras listed in [**7**]. Except for characteristic 3 his results are stronger than our Theorem 4 in [**8**]. In the present context we may apply [**8**, Theorem 4], however, to obtain

THEOREM 3. *Any semisimple restricted noncommutative Jordan algebra A of characteristic p is uniquely expressible as a direct sum $A = A_1 \oplus \cdots \oplus A_r$ of simple ideals A_i. If A is simple, then A is one of the following:*

(a) *a simple (commutative) Jordan algebra of characteristic $p \neq 2$,*

(b) *a simple flexible algebra of degree two satisfying* (3),

(c) *an algebra $B(\lambda)$ where B is a simple associative algebra over F and λ is an element ($\neq 1/2$) in the prime field P of F.*

For not all of the simple quasiassociative algebras over F need be restricted noncommutative Jordan algebras of characteristic p. Let

A be a simple quasiassociative algebra over F, and let Z be the center of A. Then A is a central simple quasiassociative algebra over Z [1, p. 586]. There exists an extension $K=Z(\lambda)$ of Z, and a central simple associative algebra B over K, such that $A_K \cong B(\lambda)$. Let R_x and L_x denote right and left multiplications in B so that the right multiplication R_x^* in $B(\lambda) \cong A_K$ is given by $R_x^* = \lambda R_x + (1-\lambda)L_x$. Then $(R_x^*)^p = R_{x^p}^*$ implies $\lambda^p R_x^p + (1-\lambda^p)L_x^p = \lambda R_{x^p} + (1-\lambda)L_{x^p}$, or $(\lambda^p - \lambda)\ (R_{x^p} - L_{x^p}) = 0$, so that either

$$\lambda^p - \lambda = 0, \tag{8}$$

or $R_{x^p} = L_{x^p}$ for every x in B. In the latter case, x^p is in the center K of the central simple associative algebra B for every x in B. But then it is well-known (for example, see [3, p. 219]) that B is commutative. In this case A is trivially included in the classification of simple algebras. Hence we may assume that (8) holds, so that λ is an element of the prime field P of characteristic p. Hence $K = Z(\lambda) = Z$, and B is a simple associative algebra over F, $A = B(\lambda)$. Conversely any such algebra satisfies (3).

A classification of the algebras (b) seems to be as complicated a matter as that of the classification of simple flexible algebras of degree two without the restriction (3). If Q is a quaternion algebra, then $Q(\lambda)$ satisfies (3) if and only if λ is in the prime field P of characteristic p, as we have seen above. Since (3) is satisfied in any alternative algebra, the algebras (b) include all Cayley algebras C. But $C(\lambda)$ satisfies (3) if and only if $\lambda \in P$ (for the "only if" part of this statement we consider the subalgebra $Q(\lambda)$ above).

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