A BOUNDARY CONDITION FOR THE VANISHING OF n HOLOMORPHIC FUNCTIONS IN COMPLEX n-SPACE

J. J. KOHN[1]

In this note we prove that *if* $f_1, \cdots, f_n$ *are holomorphic functions on* B, *the unit ball in complex n-space, and if*

$$\text{(A)} \qquad \sum_{\alpha=1}^{n} \bar{z}_\alpha f_\alpha = 0$$

on the boundary of B, *then* $f_1=0, f_2=0, \cdots, f_n=0$ *throughout* B. We assume that the f_α are continuous in the closure of B.

The above theorem can be considered as a special case of a boundary value problem for forms of type (1, 0) on a finite Kähler manifold (see [1]). Namely, let

$$\psi = \sum_{\alpha=1}^{n} f_\alpha dz_\alpha$$

and

$$\Phi = \sum_{\beta=1}^{n} z_\beta d\bar{z}_\beta;$$

then if condition (A) is satisfied and if

$$\text{(a)} \qquad \Delta(\psi \wedge \Phi) = 0$$

(where Δ is the laplacian

$$\Delta = -4 \sum_{v=1}^{n} \frac{\partial^2}{\partial z_v \partial \bar{z}_v}$$

which on differential forms acts separately on each component) it follows that ψ is zero. Note that condition (A) is a type of contraction of ψ with $\bar{\Phi}$ and that (a) is necessary and sufficient for the holomorphy of the f_α. Further note that $d\Phi$, the exterior derivative of Φ, is the form associated with the Kähler metric on B.

Proof of Theorem. Let

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$$B = \left\{(z_1, \cdots, z_n) \,\middle|\, \sum_{\alpha=1}^{n} |z_\alpha|^2 < 1\right\}$$

and

$$S = \left\{(z_1, \cdots, z_n) \,\middle|\, \sum_{\alpha=1}^{n} |z_\alpha|^2 = 1\right\}.$$

The subspace in complex n-space orthogonal to the vector $(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n)$ is spanned by the vectors

$$\begin{aligned} A_1 &= (\bar{z}_2, -\bar{z}_1, 0, \cdots, 0), \\ A_2 &= (\bar{z}_3, 0, -z_1, \cdots, 0), \\ &\vdots \\ A_{n-1} &= (\bar{z}_n, 0, \cdots, 0, -\bar{z}_1), \end{aligned}$$

so that condition (A) implies that there exist functions $\lambda_1, \cdots, \lambda_{n-1}$ on S such that

$$(f_1, \cdots, f_n) = \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_{n-1} A_{n-1}.$$

Writing this by components we get

$$\begin{aligned} f_1 &= \lambda_1 \bar{z}_2 + \lambda_2 \bar{z}_3 + \cdots + \lambda_{n-1} \bar{z}_n, \\ f_2 &= -\lambda_1 \bar{z}_1, \\ f_3 &= -\lambda_2 \bar{z}_1, \\ &\vdots \\ f_n &= -\lambda_{n-1} \bar{z}_1. \end{aligned}$$

So on S we have

$$f_\alpha = -\lambda_{\alpha-1} \bar{z}_1 \qquad \text{for } \alpha = 2, 3, \cdots, n.$$

Multiplying by z_1, we get

$$z_1 f_\alpha = -\lambda_{\alpha-1} |z_1|^2 = -\lambda_{\alpha-1}(1 - |z_2|^2 - \cdots - |z_n|^2).$$

Hence if $z_1 \neq 0$

$$\lambda_{\alpha-1} = \frac{-z_1 f_\alpha}{1 - |z_2|^2 - \cdots - |z_n|^2}.$$

Thus the λ_β can be extended to functions on $B - \{(z_1, \cdots, z_n) \mid z_1 = 0\}$ which are holomorphic in z_1. Now differentiating $\lambda_{\alpha-1}$ with respect to $\bar{z}_\beta$ we obtain

$$\frac{\partial\lambda_{\alpha-1}}{\partial\bar{z}_\beta} = \begin{cases} 0 & \text{if } \beta = 1 \\ \dfrac{-z_1 z_\beta f_\alpha}{(1 - |z_2|^2 - \cdots - |z_n|^2)^2} & \text{if } \beta > 1 \end{cases}$$

on $B - \{z | z_1 = 0\}$.

Differentiating f_1 with respect to $\bar{z}_\beta$:

$$0 = \frac{\partial\lambda_1}{\partial\bar{z}_\beta}\bar{z}_2 + \frac{\partial\lambda_2}{\partial\bar{z}_\beta}\bar{z}_3 + \cdots + \frac{\partial\lambda_{n-1}}{\partial\bar{z}_\beta}\bar{z}_n + \lambda_{\beta-1} \quad \text{for } \beta > 1.$$

Substituting the expression for $\partial\lambda_{\alpha-1}/\partial\bar{z}_\beta$ into the above equation we obtain

$$0 = \frac{-z_1 z_\beta}{(1 - |z_1|^2 - \cdots - |z_n|^2)^2}[\bar{z}_2 f_2 + \cdots + \bar{z}_n f_n] + \lambda_{\beta-1}.$$

Evaluating on S by use of condition (A)

$$z_\beta f_1 + |z_1|^2 \lambda_{\beta-1} = 0.$$

But since

$$\bar{z}_1 \lambda_{\beta-1} = -f_\beta$$

we obtain

$$z_\beta f_1 = z_1 f_\beta.$$

Multiplying by $\bar{z}_\beta$ and summing over β

$$(|z_2|^2 + |z_3|^2 + \cdots + |z_n|^2) f_1 = z_1(\bar{z}_2 f_2 + \cdots + \bar{z}_n f_n) = -|z_1|^2 f_1.$$

Adding $|z_1|^2 f_1$ to both sides and evaluating on S we get

$$f_1 = 0.$$

Similarly by appropriate choices of bases for vectors orthogonal to $(\bar{z}_1, \cdots, \bar{z}_n)$ we get $f_2 = 0, f_3 = 0, \cdots, f_n = 0$. Q.E.D.

Reference

1. J. J. Kohn and D. C. Spencer, *Complex Neumann problems*, Ann. of Math. vol. 65, no. 4 (1957).

Princeton University