E. Landau [1, §214] has given a theorem on the multiplication of Dirichlet series to the effect that if $\alpha, \beta, \rho, \tau$, are real numbers with $\min (\rho, \tau) > \max (\alpha, \beta)$ and if $\sum a_n \xi_n^{-\sigma}$ converges for $\sigma > \alpha$, absolutely for $\sigma > \rho$, $\sum b_n \xi_n^{-\tau}$ converges for $\sigma > \beta$, absolutely for $\sigma > \tau$, then the Dirichlet product of these two series converges for

$$\sigma > \frac{\sigma \tau - \alpha \beta}{\rho + \tau - \alpha - \beta}.$$  

(If $\min (\rho, \tau) \leq \max (\alpha, \beta)$ then we have convergence for $\sigma > \max (\alpha, \beta)$.) H. Bohr [2, Theorem XIX] gave an example to show that in the case $\alpha = \beta = 0$, $\rho = \tau = 1$ the above conclusion cannot be improved.

In this paper we shall use a variation of Bohr's example to give, for each $\alpha, \beta, \rho, \tau$ with $\min (\rho, \tau) > \max (\alpha, \beta)$, two Dirichlet series whose product has abscissa of convergence exactly

$$\frac{\rho \tau - \alpha \beta}{\rho + \tau - \alpha - \beta}.$$  

Thus we show that Landau's theorem is the best possible in all cases (the trivial cases being handled similarly).

Bohr [2, Theorem XVII] defines a certain Dirichlet series $\sum a_m m^{-s}$ as follows. Let $(\alpha_n)$, $(t_n)$, $(\beta_n)$, $(\gamma_n)$ be sequences of positive integers such that for all $n \geq 1$
\[ \alpha_n < \ell_n < \beta_n < \gamma_n < \alpha_{n+1}, \quad \alpha_n < (\ell_n)^{1/2}, \quad \gamma_n > \ell_n^2, \]

\[ \beta_n = \ell_n^{1+\delta_n}, \quad \text{where} \quad \lim_{n \to \infty} \delta_n = 0, \quad \lim_{n \to \infty} \ell_n^{-\delta_n} = 0. \]

For example, we could take
\[ \ell_n = 2^{2^n}, \quad \delta_n = 2^{-2^n}, \quad \beta_n = 2^{2^n+2^n}, \quad \alpha_n = 2^{2^n-1}, \quad \gamma_n = 2^{2^n+1} + 1. \]

Let \( c \) be a given positive number and define \( S_m = \sum_{j=1}^{m} a_j \) by

\[ S_m = \begin{cases} 0 & \text{for } \alpha_n \leq m < \beta_n, \\ m^{ict_n} & \text{for } \beta_n \leq m \leq \gamma_n, \\ 1 & \text{for } \gamma_n < m < \alpha_{n+1}. \end{cases} \]

(In Bohr’s original work \( c = 1 \)). Since \( |S_m| \leq 1 \) for all \( m \) and the sequence \( (S_m) \) has no limit, it is clear that the series \( \sum a_m m^{-s} \) has convergence abscissa \( 0 \). Thus the abscissa of absolute convergence is at most \( 1 \), and so \( \mu(\sigma) = 0 \) for \( \sigma \geq 1 \), where \( \mu \) is the Lindelöf function for \( f(s) = \sum a_m m^{-s} \). Bohr shows for \( c = 1, 0 < \sigma < 1 \), that \( \mu(\sigma) \geq 1 - \sigma \). If throughout Bohr’s proof we replace \( \ell_n \) by \( ct_n \) we will find that for \( 0 < \sigma_0 < 1 \), as \( n \to \infty \),

\[ f(\sigma_0 + ict_n) = \frac{ic}{\sigma_0} (\ell_n^{1-\sigma_0(1+\delta_n)} + o(\ell_n^{1-\sigma_0(1+\delta_n)})). \]

(Hence, \( \mu(\sigma) \geq 1 - \sigma \) for \( 0 < \sigma < 1 \); actually, from [1, §229], we can show, with Bohr, that \( \mu(\sigma) = 1 - \sigma \) for \( 0 < \sigma < 1 \).

Now, given \( \alpha < \rho \), take \( c = (\rho - \alpha)^{-1} \) and let

\[ g(s) = f\left( \frac{s - \alpha}{\rho - \alpha} \right) = \sum a'_m \xi'_m, \quad \text{where} \quad \xi'_m = m^{1/(\rho - \alpha)}, \quad a'_m = a_m \xi'_m^{\alpha}. \]

Then for \( \alpha < \sigma_0 < \rho \), since \( 0 < (\sigma_0 - \alpha)/(\rho - \alpha) < 1 \), by (2), as \( n \to \infty \)

\[ g(\sigma_0 + il_n) = f\left( \frac{\sigma_0 - \alpha}{\rho - \alpha} + i \frac{l_n}{\rho - \alpha} \right) = \frac{i}{\sigma_0 - \alpha} \ell_n^{1-(\sigma_0 - \alpha)/(\rho - \alpha)(1+\delta_n)} + o\left\{ \ell_n^{1-(\sigma_0 - \alpha)/(\rho - \alpha)(1+\delta_n)} \right\}. \]

Similarly, given \( \beta < \tau \), take \( c = (\tau - \beta)^{-1} \) and let

\[ h(s) = f\left( \frac{s - \beta}{\tau - \beta} \right). \]

Then for \( \beta < \sigma_0 < \tau \),
A \left( r_0 + t \beta \right) = -\frac{1}{\alpha} + O\left( \frac{1}{n} \right).

If \max (\alpha, \beta) < \sigma_0 < \min (\rho, \tau), then by (3) and (4), as \( n \to \infty \)

\[
\frac{g(\sigma_0 + it_n)h(\sigma_0 + it_n)}{2^{-(\sigma_0 - \alpha)/(\rho - \alpha) + (\sigma_0 - \beta)/(\tau - \beta)}} = \frac{1}{\alpha} + O\left( \frac{1}{n} \right).
\]

Thus the Lindelöf function for \( gh \) satisfies, since \( t_n^{-\delta_n} \to 0 \),

\[
\mu(\sigma) \geq 2 - \left\{ \frac{\sigma - \alpha}{\rho - \alpha} + \frac{\sigma - \beta}{\tau - \beta} \right\}
\]

in this interval, and so \( \mu(\sigma) \geq 1 \) for

\[
\sigma < (\rho \tau - \alpha \beta)/(\rho + \tau - \alpha - \beta).
\]

Observe that

\[
\max (\alpha, \beta) < \frac{\rho \tau - \alpha \beta}{\rho + \tau - \alpha - \beta} < \min (\rho, \tau).
\]

Therefore, by [1, §229], the Dirichlet product of \( g \) and \( h \) cannot converge if \( \sigma < (\rho \tau - \alpha \beta)/(\rho + \tau - \alpha - \beta) \), and so the abscissa of convergence is exactly \( (\rho \tau - \alpha \beta)/(\rho + \tau - \alpha - \beta) \).

Note that the above examples can also be applied to the case \( \min (\rho, \tau) \leq \max (\alpha, \beta) \).

**Bibliography**


*Ohio State University*