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THE MULTIPLICATION PROBLEM FOR DIRICHLET SERIES

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E. Landau [1, §214] has given a theorem on the multiplication of Dirichlet series to the effect that if α, β, ρ, τ, are real numbers with min $(\rho, \tau) > \max(\alpha, \beta)$ and if $\sum a_n \xi_n^{-s}$ converges for $\sigma > \alpha$, absolutely for $\sigma > \rho$, $\sum b_n \xi_n^{-s}$ converges for $\sigma > \beta$, absoutely for $\sigma > \tau$, then the Dirichlet product of these two series converges for

$$\sigma > \frac{\rho\tau - \alpha\beta}{\rho + \tau - \alpha - \beta}.$$

(If $\min(\rho, \tau) \leqq \max(\alpha, \beta)$ then we have convergence for $\sigma > \max(\alpha, \beta)$.) H. Bohr [2, Theorem XIX] gave an example to show that in the case $\alpha = \beta = 0$, $\rho = \tau = 1$ the above conclusion cannot be improved.

In this paper we shall use a variation of Bohr's example to give, for each α, β, ρ, τ with $\min(\rho, \tau) > \max(\alpha, \beta)$, two Dirichlet series whose product has abscissa of convergence exactly

$$\frac{\rho\tau - \alpha\beta}{\rho + \tau - \alpha - \beta}.$$

Thus we show that Landau's theorem is the best possible in all cases (the trivial cases being handled similarly).

Bohr [2, Theorem XVII] defines a certain Dirichlet series $\sum a_m m^{-s}$ as follows. Let (α_n), (t_n), (β_n), (γ_n) be sequences of positive integers such that for all $n \geqq 1$

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$$\alpha_n < t_n < \beta_n < \gamma_n < \alpha_{n+1}, \quad \alpha_n < (t_n)^{1/2}, \quad \gamma_n > t_n^2,$$

$$\beta_n = t_n^{1+\delta_n}, \quad \text{where} \quad \lim_n \delta_n = 0, \quad \lim_n t_n^{-\delta_n} = 0.$$

For example, we could take

$$t_n = 2^{2^{3n}}, \quad \delta_n = 2^{-2n}, \quad \beta_n = 2^{2^{3n}+2^n}, \quad \alpha_n = 2^{2^{3n-1}} - 1, \quad \gamma_n = 2^{2^{3n+1}} + 1.$$

Let c be a given positive number and define $S_m = \sum_{j=1}^{m} a_j$ by

$$S_m = \begin{cases} 0 & \text{for } \alpha_n \leqq m < \beta_n, \\ m^{ict_n} & \text{for } \beta_n \leqq m \leqq \gamma_n, \\ 1 & \text{for } \gamma_n < m < \alpha_{n+1}. \end{cases} \tag{1}$$

(In Bohr's original work $c=1$). Since $|S_m| \leqq 1$ for all m and the sequence (S_m) has no limit, it is clear that the series $\sum a_m m^{-s}$ has convergence abscissa 0. Thus the abscissa of absolute convergence is at most 1, and so $\mu(\sigma)=0$ for $\sigma \geqq 1$, where μ is the Lindelöf function for $f(s) = \sum a_m m^{-s}$. Bohr shows for $c=1$, $0<\sigma<1$, that $\mu(\sigma) \geqq 1-\sigma$. If throughout Bohr's proof we replace t_n by ct_n we will find that for $0<\sigma_0<1$, as $n \to \infty$,

$$f(\sigma_0 + ict_n) = \frac{ic}{\sigma_0} t_n^{1-\sigma_0(1+\delta_n)} + o(t_n^{1-\sigma_0(1+\delta_n)}). \tag{2}$$

(Hence, $\mu(\sigma) \geqq 1-\sigma$ for $0<\sigma<1$; actually, from [1, §229], we can show, with Bohr, that $\mu(\sigma) = 1-\sigma$ for $0<\sigma<1$).

Now, given $\alpha<\rho$, take $c=(\rho-\alpha)^{-1}$ and let

$$g(s) = f\left(\frac{s-\alpha}{\rho-\alpha}\right) = \sum a'_m \xi'^{-s}_m, \quad \text{where} \quad \xi'_m = m^{1/(\rho-\alpha)}, \quad a'_m = a_m \xi'^{\alpha}_m.$$

Then for $\alpha<\sigma_0<\rho$, since $0<(\sigma_0-\alpha)/(\rho-\alpha)<1$, by (2), as $n \to \infty$

$$\begin{aligned} g(\sigma_0 + it_n) &= f\left(\frac{\sigma_0-\alpha}{\rho-\alpha} + i\frac{t_n}{\rho-\alpha}\right) \\ &= \frac{i}{\sigma_0-\alpha} t_n^{1-(\sigma_0-\alpha)/(\rho-\alpha)(1+\delta_n)} + o\{t_n^{1-(\sigma_0-\alpha)/(\rho-\alpha)(1+\delta_n)}\}. \end{aligned} \tag{3}$$

Similarly, given $\beta<\tau$, take $c=(\tau-\beta)^{-1}$ and let

$$h(s) = f\left(\frac{s-\beta}{\tau-\beta}\right).$$

Then for $\beta<\sigma_0<\tau$,

$$(4)\quad h(\sigma_0 + it_n) = \frac{i}{\sigma_0 - \beta} t_n^{1-(\sigma_0-\beta)/(\tau-\beta)(1+\delta_n)} + o\{t_n^{1-(\sigma_0-\beta)/(\tau-\beta)(1+\delta_n)}\}.$$

If $\max(\alpha, \beta) < \sigma_0 < \min(\rho, \tau)$, then by (3) and (4), as $n \to \infty$

$$g(\sigma_0 + it_n)h(\sigma_0 + it_n) = \frac{-1}{(\sigma_0 - \alpha)(\sigma_0 - \beta)} t_n^{2-\{(\sigma_0-\alpha)/(\rho-\alpha)+(\sigma_0-\beta)/(\tau-\beta)\}(1+\delta_n)}$$

$$+ o\{t_n^{2-\{(\sigma_0-\alpha)/(\rho-\alpha)+(\sigma_0-\beta)/(\tau-\beta)\}(1+\delta_n)}\}.$$

Thus the Lindelöf function for gh satisfies, since $t_n^{-\delta n} \to 0$,

$$\mu(\sigma) \geqq 2 - \left\{\frac{\sigma - \alpha}{\rho - \alpha} + \frac{\sigma - \beta}{\tau - \beta}\right\}$$

in this interval, and so $\mu(\sigma) > 1$ for

$$\sigma < (\rho\tau - \alpha\beta)/(\rho + \tau - \alpha - \beta).$$

Observe that

$$\max(\alpha, \beta) < \frac{\rho\tau - \alpha\beta}{\rho + \tau - \alpha - \beta} < \min(\rho, \tau).$$

Therefore, by [1, §229], the Dirichlet product of g and h cannot converge if $\sigma < (\rho\tau - \alpha\beta)/(\rho + \tau - \alpha - \beta)$, and so the abscissa of convergence is exactly $(\rho\tau - \alpha\beta)/(\rho + \tau - \alpha - \beta)$.

Note that the above examples can also be applied to the case $\min(\rho, \tau) \leqq \max(\alpha, \beta)$.

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