Boas [1] has proved the following theorems:

**Theorem A.** If $\lambda_n$ is finite ultimately, and if $f(x) = \lambda_0/2 + \sum_1^\infty \lambda_n \cos nx$, then for $0 < \gamma < 1$, $x^{-\gamma}f(x) \in L(0, \pi) \iff \sum n^{\gamma-1}\lambda_n$ converges.

**Theorem B.** If $\lambda_n \geq 0$ is finite ultimately, and if $\lambda_0/2 + \sum_1^\infty \lambda_n = 0$ then (with $f(x)$ as in Theorem A) $x^{-1}f(x) \in L(0, \pi) \iff \sum (\log n)\lambda_n$ converges.

The following theorem for sine series was proved by Young [6] for $\gamma = 0$, by Boas [1] for $0 < \gamma \leq 1$, and by Heywood [4] for $1 < \gamma < 2$.

**Theorem C.** If $\lambda_n$ is finite ultimately, and if $g(x) = \sum_1^\infty \lambda_n \sin nx$, then for $0 \leq \gamma < 2$, $x^{-\gamma}g(x) \in L(0, \pi) \iff \sum n^{\gamma-1}\lambda_n$ converges.

Stronger versions for each half of Theorems A and C were proved by Boas [1] for $0 < \gamma < 1$ and for $0 < \gamma \leq 1$ respectively. For $1 < \gamma < 2$ Heywood [4] proved Theorem C when $\lambda_n \geq 0$ ultimately.

Heywood [4] also has proved the following extension of Theorems A and B.

**Theorem D.** If $\lambda_n \geq 0$ is finite ultimately, and if $\lambda_0/2 + \sum_1^\infty \lambda_n = 0$, then for $1 < \gamma < 3$ $x^{-1}f(x) \in L(0, \pi) \iff \sum n^{\gamma-1}\lambda_n$ converges.

By using a result of Hartman and Wintner [3], Heywood [4] showed that for $\gamma \geq 3$ and for $\gamma \geq 2$, respectively, Theorem D and Theorem C break down. On the other hand Boas and González-Fernández [2] have proved the following theorem, proved before by Heywood [4] for $\gamma < 2$.

**Theorem E.** If $h(x) = \sum_1^\infty \lambda_n x^n$ has radius of convergence 1, if $\lambda_n \geq 0$ ultimately, and if $\gamma < 1$ or if $k \leq \gamma < k+1$ (where $k$ is a positive integer) then provided that

$$\sum_0^\infty \lambda_n = \sum_1^\infty n\lambda_n = \cdots = \sum_{k-1}^\infty n(n - 1) \cdots (n - k + 2)\lambda_n = 0$$

(i) for $\gamma \neq k$, $(1-x)^{-\gamma}h(x) \in L(0, 1) \iff \sum n^{\gamma-1}\lambda_n$ converges,

(ii) for $\gamma = k$, $(1-x)^{-\gamma}h(x) \in L(0, 1) \iff \sum n^{\gamma-1}(\log n)\lambda_n$ converges.

The structure of Theorem E suggests companion theorems for the
cosine and for the sine theorems. In this note we shall prove the following theorems:

**Theorem 1.** If \( \lambda_n \geq 0 \) ultimately let \( \lambda_0/2 + \sum_1^\infty \lambda_n \cos nx \) converge to \( f(x) \); if for some integer \( j \geq 1 \),

\[
\frac{1}{2} \lambda_0 + \sum_1^\infty \lambda_n = \sum_1^\infty n^2 \lambda_n = \cdots = \sum_1^\infty n^{2j} \lambda_n = 0
\]

then (i) for \( 2j+1 < \gamma < 2(j+1) + 1 \), \( x^{-\gamma}f(x) \in L(0, \pi) \) if \( \sum n^{\gamma-1} \lambda_n \) converges, and (ii) for \( \gamma = 2j+1 \), \( x^{-\gamma}f(x) \in L(0, \pi) \) if \( \sum n^{\gamma-1} (\log n) \lambda_n \) converges.

**Theorem 2.** If \( \lambda_n \geq 0 \) ultimately let \( \sum_1^\infty \lambda_n \sin nx \) converge to \( g(x) \); if for some integer \( l \geq 1 \),

\[
\sum_1^\infty n \lambda_n = \sum_1^\infty n^3 \lambda_n = \cdots = \sum_1^\infty n^{2l-1} \lambda_n = 0
\]

then (i) for \( 2l < \gamma < 2(l+1) \), \( x^{-\gamma}g(x) \in L(0, \pi) \) if \( \sum n^{\gamma-1} \lambda_n \) converges, and (ii) for \( \gamma = 2l \), \( x^{-\gamma}g(x) \in L(0, \pi) \) if \( \sum n^{\gamma-1} (\log n) \lambda_n \) converges.

Before going into the proof of Theorem 1, let us examine the nature of the assumption (1). We have that if \( \sum n^{\gamma-1} (\log n) \lambda_n \) converges for \( \gamma = 2j+1 \) or if \( \sum n^{\gamma-1} \lambda_n \) converges for \( 2j+1 < \gamma < 2(j+1) + 1 \) then the series \( \lambda_0/2 + \sum_1^\infty \lambda_n \), \( \sum_1^\infty n^2 \lambda_n \), \( \cdots \), \( \sum_1^\infty n^{2j} \lambda_n \) converge; but \( \sum_1^\infty n^{2k} \lambda_n \cos nx = (-1)^k f^{(2k)}(x) \), \( 0 \leq k \leq j \), therefore (by uniform convergence) \( f^{(2k)}(x) \to \sum_1^\infty n^{2k} \lambda_n \) as \( x \to 0 \) for \( 0 \leq k \leq j \). On the other hand

\[
f(x) = f(0) + xf^{(1)}(x) + \frac{x^2}{2!} f^{(2)}(x) + \cdots + \frac{x^{2j-1}}{(2j-1)!} f^{(2j-1)}(x)
\]

\[+ \frac{x^{2j}}{(2j)!} f^{(2j)}(\theta x), \]

\[0 \leq \theta \leq 1,\]

and

\[
x^{-\gamma}f(x) = x^{-\gamma}f(0) + x^{-\gamma+1}f^{(1)}(x) + \cdots + \frac{x^{-\gamma+2j}}{(2j)!} f^{(2j)}(\theta x)
\]

from which it follows that in order for \( x^{-\gamma}f(x) \in L(0, \pi) \) we must have that

\[
\frac{1}{2} \lambda_0 + \sum_1^\infty \lambda_n = \sum_1^\infty n^2 \lambda_n = \cdots = \sum_1^\infty n^{2j} \lambda_n = 0.
\]

An analogous comment applies to assumption (2) of Theorem 2.
For the proof of Theorem 1 we need the following simple lemma:

**Lemma 1.**

\[
\cos y - 1 + \frac{y^2}{2!} - \cdots + (-1)^j \frac{y^{2j+1}}{(2j)!} \leq 0 \quad \text{for } j \text{ even,}
\]

\[
\cos y - 1 + \frac{y^2}{2!} - \frac{y^4}{4!} + \cdots - \frac{y^{2j}}{(2j)!} \geq 0 \quad \text{for } j \text{ odd.}
\]

**Proof.** We have that \(\cos y = 1 - \frac{y^2}{2!} + \cdots + (-1)^j \frac{y^{2j}}{(2j)!} + \cdots\). For \(j\) even write

\[
\cos y - 1 = \frac{y^2}{2!} - \frac{y^4}{4!} + \cdots - \frac{y^{2j}}{(2j)!} = -\sum_{i=0}^{j} \frac{y^{2i+1}}{(2i+1)!} + \frac{y^{2(j+1)}}{2(j+1)!} - \cdots
\]

by repeated differentiation of the right hand side of (3) we change it into \(-\sin y\); the process is legitimate because of the uniform convergence of the differentiated series; then by integrating the last series obtained, from 0 to \(y\), repeatedly, legitimate by uniform convergence, we get back the right hand side of (3), hence it is not positive.

For \(j\) odd write

\[
\cos y - 1 + \frac{y^2}{2!} - \frac{y^4}{4!} + \cdots - \frac{y^{2j}}{(2j)!} = -\frac{y^{2(j+1)}}{[2(j+1)]!} + \frac{y^{2(j+2)}}{[2(j+2)]!} - \cdots
\]

an analogous process yields that (4) is not negative.

We now prove Theorem 1. By using (1) we can write

\[
f(x) = \sum_{i=0}^{\infty} \lambda_n \left[ \cos nx - 1 + \frac{(nx)^2}{2!} - \cdots + (-1)^{i+1} \frac{(nx)^{2j}}{(2j)!} \right] = \sum_{i=0}^{\infty} \lambda_n K_i(nx).
\]

By Lemma 1, for every \(n\) and \(x\), \(K_i(nx)\) is of the same sign. If \(j\) is even we write \(-f(x) = \sum_{i=0}^{\infty} \lambda_n (-K_i(nx))\) where \(-K_i(nx) \geq 0\); if \(j\) is odd we write \(f(x) = \sum_{i=0}^{\infty} \lambda_n K_i(nx)\) where \(K_i(nx) \geq 0\).

Since

\[
K_i(nx) = (-1)^{i+1} \frac{(nx)^{2(i+1)}}{[2(j+1)]!} + (-1)^{i+2} \frac{(nx)^{2(i+2)}}{[2(j+2)]!} + \cdots
\]

then for \(x \to 0\) and fixed \(n\), \(K_i(nx) \sim Ax^{2(i+1)}\), hence for \(2(j+1) < \gamma < 2(j+1) + 1\) we have that for every \(n\), \(x^{-\gamma}K_i(nx) \in L(0, \pi)\).
For the sake of definiteness suppose that $j$ is odd, and suppose that for $n \geq N$, $\lambda_n \geq 0$; then write
\[
\int_0^\pi x^{-\gamma} f(x) dx = \int_0^\pi x^{-\gamma} \sum_{n=1}^\infty \lambda_n K_j(nx) dx
= \int_0^\pi x^{-\gamma} \left( \sum_{n=1}^{N-1} \lambda_n + \sum_{n=N}^\infty \right) K_j(nx) dx.
\]

Since for every $n$, $x^{-\gamma} K_j(nx) \in L(0, \pi)$ then $x^{-\gamma} \sum_{n=1}^{N-1} \lambda_n K_j(nx) \in L(0, \pi)$, therefore $x^{-\gamma} f(x) \in L(0, \pi) \iff x^{-\gamma} \sum_{n=N}^\infty \lambda_n K_j(nx) \in L(0, \pi)$, but since the latter is a series of positive terms then
\[
x^{-\gamma} \sum_{n=N}^\infty \lambda_n K_j(nx) \in L(0, \pi) \iff \sum_{n=N}^\infty \lambda_n \int_0^\pi x^{-\gamma} K_j(nx) dx
= \sum_{n=N}^\infty \lambda_n n^{\gamma-1} \int_0^{n\pi} y^{-\gamma} K_j(y) dy.
\]

Converges.

Now if $\gamma = 2j+1$
\[
\int_0^{n\pi} y^{-(2j+1)} K_j(y) dy
= \int_0^{n\pi} y^{-\gamma} \left[ \cos y - 1 + \frac{y^2}{2!} - \cdots + \frac{y^{2j}}{(2j)!} \right] dy \sim \frac{1}{(2j)!} \log n
\]
so by positivity we have that (5) converges $\iff \sum_{n=1}^\infty n^{\gamma-1} (\log n) \lambda_n$ converges, which proves (i) for $j$ odd.

Consider now the case $2j+1 < \gamma < 2(j+1)+1$. Since we assume $j$ to be odd, $K_j(y) \geq 0$, therefore $\int_0^{n\pi} y^{-\gamma} K_j(y) dy$ is $\geq 0$ and $\uparrow$ with $n$, hence
\[
\sum_{n=1}^\infty \lambda_n n^{\gamma-1} \int_0^{n\pi} y^{-\gamma} K_j(y) dy \leq \sum_{n=1}^\infty \lambda_n n^{\gamma-1} \int_0^{n\pi} y^{-\gamma} K_j(y) dy.
\]

Hence if (5) converges then $\sum_{n=1}^\infty \lambda_n n^{\gamma-1}$ does so, which proves the "only if" part of (ii) for $j$ odd.

Since
\[
K_j(y) = \left( \cos y - 1 + \frac{y^2}{2!} - \cdots + \frac{y^{2j}}{(2j)!} \right) \sim \frac{1}{(2j)!} y^{2j} \text{ as } y \to \infty
\]
then
\[
y^{-\gamma} K_j(y) \sim \frac{1}{(2j)!} y^{-\gamma+2j} \text{ as } y \to \infty,
\]
and since $1 < \gamma - 2j$ then $\int_0^\infty y^{-\gamma} K_j(y) dy < \infty$. Now write
\[
\sum_{N} \lambda_n n^{\gamma-1} \int_0^\infty y^{-\gamma} K_j(y) dy \leq \sum_{N} \lambda_n n^{\gamma-1} \int_0^\infty y^{-\gamma} K_j(y) dy
\]
so the convergence of $\sum \lambda_n n^{\gamma-1}$ implies the convergence of (5), thus the "if" part of (ii) is proved, for $j$ odd.

For $j$ even the proof is mutatis mutandis the same.

The proof of Theorem 2 is analogous to the proof of Theorem 1; the role of Lemma 1 is taken here by the following lemma.

**Lemma 2.**
\[
sin y - y + \frac{y^3}{3!} - \frac{y^5}{5!} + \cdots + (-1)^l \frac{y^{2l-1}}{(2l-1)!} \geq 0 \quad \text{for } l \text{ even},
\]
\[
sin y - y + \frac{y^3}{3!} - \frac{y^5}{5!} + \cdots + (-1)^l \frac{y^{2l-1}}{(2l-1)!} \leq 0 \quad \text{for } l \text{ odd}.
\]

The proof of Lemma 2 is analogous to the proof of Lemma 1, or shorter:
\[
\frac{d}{dy} \left[ \sin y - y + \frac{y^3}{3!} - \cdots + (-1)^l \frac{y^{2l-1}}{(2l-1)!} \right] = \cos y - 1 + \frac{y^2}{2!} - \cdots + (-1)^l \frac{y^{2(l-1)}}{[2(l-1)!]}
\]
by Lemma 1 the right hand side has a fixed sign, so the lemma follows because $\sin y - y + \cdots + (-1)^l y^{2l-1}/(2l-1)!$ is 0 at $y = 0$.

**References**