INTEGRABILITY OF TRIGONOMETRIC SERIES[1]

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Boas [1] has proved the following theorems:

THEOREM A. *If* $\lambda_n \downarrow 0$ *ultimately, and if* $f(x) = \lambda_0/2 + \sum_1^\infty \lambda_n \cos nx$, *then for* $0 < \gamma < 1$, $x^{-\gamma} f(x) \in L(0, \pi) \Leftrightarrow \sum n^{\gamma-1}\lambda_n$ *converges.*

THEOREM B. *If* $\lambda_n \geqq 0$ *ultimately, and if* $\lambda_0/2 + \sum_1^\infty \lambda_n = 0$ *then (with* $f(x)$ *as in Theorem A)* $x^{-1} f(x) \in L(0, \pi) \Leftrightarrow \sum (\log n)\lambda_n$ *converges.*

The following theorem for sine series was proved by Young [6] for $\gamma = 0$, by Boas [1] for $0 < \gamma \leqq 1$, and by Heywood [4] for $1 < \gamma < 2$.

THEOREM C. *If* $\lambda_n \downarrow 0$ *ultimately, and if* $g(x) = \sum_1^\infty \lambda_n \sin nx$, *then for* $0 \leqq \gamma < 2$, $x^{-\gamma} g(x) \in L(0, \pi) \Leftrightarrow \sum n^{\gamma-1}\lambda_n$ *converges.*

Stronger versions for each half of Theorems A and C were proved by Boas [1] for $0 < \gamma < 1$ and for $0 < \gamma \leqq 1$ respectively. For $1 < \gamma < 2$ Heywood [4] proved Theorem C when $\lambda_n \geqq 0$ ultimately.

Heywood [4] also has proved the following extension of Theorems A and B.

THEOREM D. *If* $\lambda_n \geqq 0$ *ultimately, and if* $\lambda_0/2 + \sum_1^\infty \lambda_n = 0$, *then for* $1 < \gamma < 3$ $x^{-\gamma} f(x) \in L(0, \pi) \Leftrightarrow \sum n^{\gamma-1}\lambda_n$ *converges.*

By using a result of Hartman and Wintner [3], Heywood [4] showed that for $\gamma \geqq 3$ and for $\gamma \geqq 2$, respectively, Theorem D and Theorem C break down. On the other hand Boas and González-Fernández [2] have proved the following theorem, proved before by Heywood [4] for $\gamma < 2$.

THEOREM E. *If* $h(x) = \sum_0^\infty \lambda_n x^n$ *has radius of convergence* 1, *if* $\lambda_n \geqq 0$ *ultimately, and if* $\gamma < 1$ *or if* $k \leqq \gamma < k+1$ *(where* k *is a positive integer) then provided that*

$$\sum_0^\infty \lambda_n = \sum_1^\infty n\lambda_n = \cdots = \sum_{k-1}^\infty n(n-1)\cdots(n-k+2)\lambda_n = 0$$

(i) *for* $\gamma \neq k$, $(1-x)^{-\gamma} h(x) \in L(0, 1) \Leftrightarrow \sum n^{\gamma-1}\lambda_n$ *converges,*
(ii) *for* $\gamma = k$, $(1-x)^{-\gamma} h(x) \in L(0, 1) \Leftrightarrow \sum n^{\gamma-1}(\log n)\lambda_n$ *converges.*

The structure of Theorem E suggests companion theorems for the

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cosine and for the sine theorems. In this note we shall prove the following theorems:

THEOREM 1. *If* $\lambda_n \geqq 0$ *ultimately let* $\lambda_0/2 + \sum_1^\infty \lambda_n \cos nx$ *converge to* $f(x)$; *if for some integer* $j \geqq 0$,

$$(1) \qquad \frac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n = \sum_1^\infty n^2\lambda_n = \cdots = \sum_1^\infty n^{2j}\lambda_n = 0$$

then (i) *for* $2j+1 < \gamma < 2(j+1)+1$, $x^{-\gamma}f(x) \in L(0, \pi) \Leftrightarrow \sum n^{\gamma-1}\lambda_n$ *converges, and* (ii) *for* $\gamma = 2j+1$, $x^{-\gamma}f(x) \in L(0, \pi) \Leftrightarrow \sum n^{\gamma-1}(\log n)\lambda_n$ *converges.*

THEOREM 2. *If* $\lambda_n \geqq 0$ *ultimately let* $\sum_1^\infty \lambda_n \sin nx$ *converge to* $g(x)$; *if for some integer* $l \geqq 1$,

$$(2) \qquad \sum_1^\infty n\lambda_n = \sum_1^\infty n^3\lambda_n = \cdots = \sum_1^\infty n^{2l-1}\lambda_n = 0$$

then (i) *for* $2l < \gamma < 2(l+1)$, $x^{-\gamma}g(x) \in L(0, \pi) \Leftrightarrow \sum n^{\gamma-1}\lambda_n$ *converges, and* (ii) *for* $\gamma = 2l$, $x^{-\gamma}g(x) \in L(0, \pi) \Leftrightarrow \sum n^{\gamma-1}(\log n)\lambda_n$ *converges.*

Before going into the proof of Theorem 1, let us examine the nature of the assumption (1). We have that if $\sum n^{\gamma-1}(\log n)\lambda_n$ converges for $\gamma = 2j+1$ or if $\sum n^{\gamma-1}\lambda_n$ converges for $2j+1 < \gamma < 2(j+1)+1$ then the series $\lambda_0/2 + \sum_1^\infty \lambda_n$, $\sum_1^\infty n^2\lambda_n, \cdots, \sum_1^\infty n^{2j}\lambda_n$ converge; but $\sum_1^\infty n^{2k}\lambda_n \cos nx = (-1)^k f^{(2k)}(x)$, $0 \leqq k \leqq j$, therefore (by uniform convergence) $f^{(2k)}(x) \to \sum_1^\infty n^{2k}\lambda_n$ as $x \to 0$ for $0 \leqq k \leqq j$. On the other hand

$$f(x) = f(0) + xf^{(1)}(x) + \frac{x^2}{2!}f^{(2)}(x) + \cdots + \frac{x^{2j-1}}{(2j-1)!}f^{(2j-1)}(x) + \frac{x^{2j}}{(2j)!}f^{(2j)}(\theta x),$$

$0 \leqq \theta \leqq 1$, and

$$x^{-\gamma}f(x) = x^{-\gamma}f(0) + x^{-\gamma+1}f^{(1)}(x) + \cdots + \frac{x^{-\gamma+2j}}{(2j)!}f^{(2j)}(\theta x)$$

from which it follows that in order for $x^{-\gamma}f(x) \in L(0, \pi)$ we must have that

$$(1) \qquad \frac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n = \sum_1^\infty n^2\lambda_n = \cdots = \sum_1^\infty n^{2j}\lambda_n = 0.$$

An analogous comment applies to assumption (2) of Theorem 2.

For the proof of Theorem 1 we need the following simple lemma:

LEMMA 1.

$$\cos y - 1 + \frac{y^2}{2!} - \cdots + (-1)^j \frac{y^{2j+1}}{(2j)!} \begin{cases} \leqq 0 & \text{for } j \text{ even}, \\ \geqq 0 & \text{for } j \text{ odd}. \end{cases}$$

PROOF. We have that $\cos y = 1 - y^2/2! + \cdots + (-1)^j y^{2j}/(2j)! + \cdots$. For j even write

$$\cos y - 1 + \frac{y^2}{2!} - \frac{y^4}{4!} + \cdots - \frac{y^{2j}}{(2j)!} \tag{3}$$
$$= - \frac{y^{2(j+1)}}{[2(j+1)]!} + \frac{y^{2(j+2)}}{[2(j+2)]!} - \cdots$$

by repeated differentiation of the right hand side of (3) we change it into $-\sin y$; the process is legitimate because of the uniform convergence of the differentiated series; then by integrating the last series obtained, from 0 to y, repeatedly, legitimate by uniform convergence, we get back the right hand side of (3), hence it is not positive.

For j odd write

$$\cos y - 1 + \frac{y^2}{2!} - \cdots + \frac{y^{2j}}{(2j)!} = \frac{y^{2(j+1)}}{[2(j+1)]!} - \cdots \tag{4}$$

an analogous process yields that (4) is not negative.

We now prove Theorem 1. By using (1) we can write

$$f(x) = \sum_1^\infty \lambda_n \left[\cos nx - 1 + \frac{(nx)^2}{2!} - \cdots + (-1)^{j+1} \frac{(nx)^{2j}}{(2j)!} \right]$$
$$= \sum_1^\infty \lambda_n K_j(nx).$$

By Lemma 1, for every n and x, $K_j(nx)$ is of the same sign. If j is even we write $-f(x) = \sum_1^\infty \lambda_n(-K_j(nx))$ where $-K_j(nx) \geqq 0$; if j is odd we write $f(x) = \sum_1^\infty \lambda_n K_j(nx)$ where $K_j(nx) \geqq 0$.

Since

$$K_j(nx) = (-1)^{j+1} \frac{(nx)^{2(j+1)}}{[2(j+1)]!} + (-1)^{j+2} \frac{(nx)^{2(j+2)}}{[2(j+2)]!} + \cdots$$

then for $x \to 0$ and fixed n, $K_j(nx) \sim Ax^{2(j+1)}$, hence for $2(j+1) < \gamma < 2(j+1)+1$ we have that for every n, $x^{-\gamma} K_j(nx) \in L(0, \pi)$.

For the sake of definiteness suppose that j is odd, and suppose that for $n \geqq N$, $\lambda_n \geqq 0$; then write

$$\int_0^\pi x^{-\gamma} f(x)dx = \int_0^\pi x^{-\gamma} \sum_1^\infty \lambda_n K_j(nx)dx$$
$$= \int_0^\pi x^{-\gamma} \left(\sum_1^{N-1} + \sum_N^\infty \right) K_j(nx)dx.$$

Since for every n, $x^{-\gamma} K_j(nx) \in L(0, \pi)$ then $x^{-\gamma} \sum_1^{N-1} \lambda_n K_j(nx) \in L(0, \pi)$, therefore $x^{-\gamma} f(x) \in L(0, \pi) \Leftrightarrow x^{-\gamma} \sum_N^\infty \lambda_n K_j(nx) \in L(0, \pi)$, but since the latter is a series of positive terms then

$$x^{-\gamma} \sum_N^\infty \lambda_n K_j(nx) \in L(0, \pi) \Leftrightarrow \sum_N^\infty \lambda_n \int_0^\pi x^{-\gamma} K_j(nx)dx$$
$$(5) \qquad = \sum_N^\infty \lambda_n n^{\gamma-1} \int_0^{n\pi} y^{-\gamma} K_j(y)dy$$

converges.

Now if $\gamma = 2j+1$

$$\int_0^{n\pi} y^{-(2j+1)} K_j(y)dy$$
$$= \int_0^{n\pi} y^{-\gamma} \left[\cos y - 1 + \frac{y^2}{2!} - \cdots + \frac{y^{2j}}{(2j)!} \right] dy \sim \frac{1}{(2j)!} \log n$$

so by positivity we have that (5) converges $\Leftrightarrow \sum_N^\infty n^{\gamma-1}(\log n)\lambda_n$ converges, which proves (i) for j odd.

Consider now the case $2j+1 < \gamma < 2(j+1)+1$. Since we assume j to be odd, $K_j(y) \geqq 0$, therefore $\int_0^{n\pi} y^{-\gamma} K_j(y)dy$ is $\geqq 0$ and $\uparrow$ with n, hence

$$\sum_N^\infty \lambda_n n^{\gamma-1} \int_0^{N\pi} y^{-\gamma} K_j(y)dy \leqq \sum_N^\infty \lambda_n n^{\gamma-1} \int_0^{n\pi} y^{-\gamma} K_j(y)dy.$$

Hence if (5) converges then $\sum \lambda_n n^{\gamma-1}$ does so, which proves the "only if" part of (ii) for j odd.

Since

$$K_j(y) = \left(\cos y - 1 + \frac{y^2}{2!} - \cdots + \frac{y^{2j}}{(2j)!} \right) \sim \frac{1}{(2j)!} y^{2j} \text{ as } y \to \infty$$

then

$$y^{-\gamma} K_j(y) \sim \frac{1}{(2j)!} y^{-\gamma+2j} \text{ as } y \to \infty,$$

and since $1 < \gamma - 2j$ then $\int_0^\infty y^{-\gamma} K_j(y)dy < \infty$. Now write

$$\sum_N^\infty \lambda_n n^{\gamma-1} \int_0^{n\pi} y^{-\gamma} K_j(y)dy \leqq \sum_N^\infty \lambda_n n^{\gamma-1} \int_0^\infty y^{-\gamma} K_j(y)dy$$

so the convergence of $\sum \lambda_n n^{\gamma-1} \Rightarrow$ the convergence of (5), thus the "if" part of (ii) is proved, for j odd.

For j even the proof is mutatis mutandis the same.

The proof of Theorem 2 is analogous to the proof of Theorem 1; the role of Lemma 1 is taken here by the following lemma.

LEMMA 2.

$$\sin y - y + \frac{y^3}{3!} - \frac{y^5}{5!} + \cdots + (-1)^l \frac{y^{2l-1}}{(2l-1)!} \begin{cases} \geqq 0 & \textit{for } l \textit{ even,} \\ \leqq 0 & \textit{for } l \textit{ odd.} \end{cases}$$

The proof of Lemma 2 is analogous to the proof of Lemma 1, or shorter:

$$\frac{d}{dy}\left[\sin y - y + \frac{y^3}{3!} - \cdots + (-1)^l \frac{y^{2l-1}}{(2l-1)!}\right]$$
$$= \cos y - 1 + \frac{y^2}{2!} - \cdots + (-1)^l \frac{y^{2(l-1)}}{[2(l-1)]!}$$

by Lemma 1 the right hand side has a fixed sign, so the lemma follows because $\sin y - y + \cdots + (-1)^l y^{2l-1}/(2l-1)!$ is 0 at $y=0$.

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