ON A THEOREM BY A. E. TAYLOR

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Let B be a complex normed linear space. It is well known [1] that, for any bounded linear functional ϕ defined on a linear subspace M of B, there exists a norm-preserving linear extension f of ϕ to B, i.e. a bounded linear functional f defined on B such that (i) $f(x) = \phi(x)$ for all $x \in M$, (ii) $\|f\|_B = \|\phi\|_M$, where $\|f\|_B$ and $\|\phi\|_M$ denote the norms of bounded linear functionals f and ϕ on B and M, respectively. It was proved by A. E. Taylor [2] that if the conjugate space of B is strictly convex, then f is uniquely determined by ϕ. The purpose of this note is to show that the converse of this theorem is true, i.e. we want to prove the following

THEOREM. *Let B be a complex normed linear space whose conjugate space is not strictly convex. Then there exists a bounded linear functional defined on a linear subspace of B for which a norm preserving linear extension to B is not unique.*

PROOF. Let f_1 and f_2 be two bounded linear functionals on B such that (i) $f_1 \neq f_2$, (ii) $\|f_1\|_B = \|f_2\|_B = \|(f_1+f_2)/2\|_B = 1$. Let us put $M = \{x \mid f_1(x) = f_2(x)\}$ and $\phi(x) = f_1(x) = f_2(x)$ on M. It suffices to prove that $\|\phi\|_M = 1$. Let z be an element of B such that $f_1(z) - f_2(z) = 1$. Then every element x of B can be uniquely expressed in the form: $x = y + az$, where $y \in M$ and $a = f_1(x) - f_2(x)$ is a complex number. Let $\{x_n \mid n = 1, 2, \cdots\}$ be a sequence of elements of B such that $\|x_n\| = 1$ for $n = 1, 2, \cdots$ and $\lim_{n\to\infty} (f_1(x_n) + f_2(x_n))/2 = 1$. Then it is easy to see that $\lim_{n\to\infty} f_1(x_n) = \lim_{n\to\infty} f_2(x_n) = 1$. Thus, if we put $x_n = y_n + a_n z$, where $y_n \in M$ and $a_n = f_1(x_n) - f_2(x_n)$, $n = 1, 2, \cdots$, then $\lim_{n\to\infty} a_n = 0$, and hence

$$\lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \phi(y_n) = \lim_{n\to\infty} f_1(y_n) = 1.$$

From this follows that $\|\phi\|_M \geqq 1$ and hence $\|\phi\|_M = 1$.

REFERENCES

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2. A. E. Taylor, *The extension of linear functionals*, Duke Math. J. vol. 5 (1939) pp. 538–547.

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