

EXISTENCE AND UNIQUENESS OF FLOWS BEHIND THREE-DIMENSIONAL STATIONARY AND PSEUDO-STATIONARY SHOCKS¹

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1. **Introduction.** Recently we have discussed the flows immediately behind three-dimensional shock waves in steady [1; 2] as well as pseudo-stationary [3] gas flows. In that analysis certain shock surfaces were found behind which the flow remains irrotational. The purpose of this paper is to show that the integration of the various conservation equations is equivalent to the solution of a Cauchy problem with the shock front as surface on which the initial data is given. By Cauchy-Kowaleski theorem it is proved that in a neighborhood of the shock the flow behind the shock exists and is uniquely determined. The corresponding case for two-dimensional gas flows has been analysed by Taub [4].

The paper has been divided into two parts. First part deals with the stationary flows, while the second part deals with the pseudo-stationary flows. The detailed analysis is given only for the stationary flows. Formal methods of tensor analysis have been used throughout the paper.

PART I. STATIONARY FLOWS

2. **Equations of motion and shock conditions.** We consider the equations governing the flow of a stationary ideal gas with viscosity and thermal-conductivity zero, namely

$$(1) \quad \rho_{,i}u_i + \rho u_{j,j} = 0, \text{ (equation of continuity),}$$

$$(2) \quad p_{,i} + \rho u_j u_{i,j} = 0, \text{ (equations of motion),}$$

$$(3) \quad S_{,i}u_i = 0, \text{ (changes of state are adiabatic),}$$

in which p , ρ , u_i and S denote the pressure, density, velocity components and entropy respectively. We assume the motion referred to a system of rectangular coordinates x^i .

If a shock wave arises, it will be assumed that this divides the flow under consideration into a region 1 and a region 2. We suppose the region 1 to be traversed by the gas before contact with the shock wave

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and region 2 to be that in which the gas enters after passing through it. Denote by u_{1i} , ρ_1 and p_1 the velocity components, density, and pressure on the wave bordering region 1 and correspondingly by u_{2i} , ρ_2 and p_2 the values of these quantities on the side of the wave bordering the region 2. Put

$$u_{1n} = u_{1i}n_i; \quad u_{2n} = u_{2i}n_i,$$

where n_i are the components of the unit normal to the shock surface, directed from region 1 into region 2. Then the Rankine-Hugoniot shock conditions for the stationary wave under consideration can be written in the form

$$(4) \quad [u_i] = \frac{-\delta u_{1n}n_i}{1 + \delta}, \quad [p] = \frac{\delta \rho_1 u_{1n}^2}{1 + \delta}, \quad [\rho] = \delta \rho_1,$$

where

$$(5) \quad \delta = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma - 1)\rho_1 u_{1n}^2},$$

and the bracket [] denotes the difference of the values on the two sides of the shock surface of the quantity enclosed. In the subsequent discussion, for simplicity, we shall omit the subscript 2 on the quantities appearing in the left members of these equations.

3. The new coordinate system. The shock configuration in a three-dimensional flow may be represented by the equations [3]:

$$(6) \quad x_i = a_i(y_1, y_2), \quad i = 1, 2, 3,$$

where y_α ($\alpha = 1, 2$) are the Gaussian coordinates on the shock surface.

Extending Taub's analysis [4], we define a transformation $x_i = a_i(y_1, y_2, T)$, from the variables x_i to a new set of variables y_α and T . T denotes the arc length of the streamlines behind the shock surface and a_i are the solutions of the equations

$$(7) \quad \frac{\partial a_i}{\partial T} = \frac{u_i}{v}, \quad v^2 = u_i u_i,$$

with the initial condition

$$(8) \quad a_i(y_1, y_2, 0) = a_i(y_1, y_2).$$

The functions on the right hand side of this equation are the functions entering equations (6). Let us further define

$$(9) \quad \partial a_i / \partial y^\alpha = A_{i\alpha},$$

$$(10) \quad A_{i\alpha}(y_1, y_2, 0) = \alpha_{i\alpha},$$

where $a_{i\alpha}a_{i\beta} = g_{\alpha\beta}$ are the components of the first fundamental form of the shock surface [5].

From (7) and (9) we have

$$(11) \quad \frac{\partial A_{i\alpha}}{\partial T} = \frac{\partial}{\partial y^\alpha} \left(\frac{u_i}{v} \right).$$

Also

$$(12) \quad A_{i\alpha,\beta} = B_{\alpha\beta} N_i / N,$$

where

$$(13) \quad N_i = \frac{1}{2} \epsilon^{\alpha\beta} e_{ijk} A_{j\alpha} A_{k\beta}; \quad N^2 = N_i N_i,$$

$$N_i(y_1, y_2, 0) = n_i(y_1, y_2); \quad A_{i\alpha} N_i = 0,$$

and $B_{\alpha\beta}$ are the components of the second fundamental form of the surface and $\epsilon^{\alpha\beta}$ and e_{ijk} are the components of the surface and space permutation tensors, respectively (5).

In view of the above mentioned transformation we may write for any function

$$(14) \quad f_{,i} = \frac{\partial f}{\partial T} T_{,i} + \frac{\partial f}{\partial y^\alpha} y_{,i}^\alpha.$$

Putting $f = a_i$ and substituting from (7) and (9) we have

$$(15) \quad \delta_{ij} = \frac{u_i}{v} T_{,j} + A_{i\alpha} y_{,j}^\alpha.$$

Multiplying both sides of (15) by N_i and $2^{-1}\epsilon^{\alpha\beta}e_{ijk}u_iA_{k\beta}$ in turn, making use of the geometrical relations (13) and the skew-symmetry of the permutations tensors, we get

$$(16) \quad T_{,j} = \frac{v}{u_n} N_j,$$

and

$$(17) \quad y_{,j}^\alpha = \frac{1}{2u_n} \epsilon^{\alpha\beta} e_{ijk} A_{k\beta} u_i,$$

where

$$(18) \quad u_n = u_i N_i.$$

Thus

$$(19) \quad u_{i,j} = \frac{vN_j}{u_n} \frac{\partial u_i}{\partial T} + y_{,j}^\alpha \frac{\partial u_i}{\partial y^\alpha},$$

$$(20) \quad u_{i,j}u_j = v \frac{\partial u^i}{\partial T}.$$

Furthermore it follows from equations (16) and (17) that for an arbitrary function f we have

$$(21) \quad u_i f_{,i} = v \frac{\partial f}{\partial T}.$$

Now with the help of the equations (19) and (21) the equations (1) to (3) are transformed respectively to

$$(22) \quad \frac{\partial}{\partial T} (\rho u_n) = 0,$$

$$(23) \quad \frac{\partial u_i}{\partial T} + \frac{N_i}{\rho u_n} \frac{\partial p}{\partial T} = \frac{1}{\rho v} \frac{\partial p}{\partial y^\alpha} y_{,i}^\alpha,$$

and

$$(24) \quad v \frac{\partial}{\partial T} (p\rho^{-\gamma}) = 0.$$

Noting that $u_n = 2^{-1} \epsilon^{\alpha\beta} e_{ijk} u_i A_{j\alpha} A_{k\beta}$, and making use of the relation (11) and the skew-symmetry of the quantities $\epsilon^{\alpha\beta}$ and e_{ijk} , the equation (22) reduces to

$$(25) \quad \frac{u_n}{\rho} \frac{\partial \rho}{\partial T} + N_i \frac{\partial u_i}{\partial T} = - \frac{\epsilon^{\alpha\beta} e_{ijk} u_i A_{k\beta}}{v} \frac{\partial u_j}{\partial y^\alpha}.$$

Furthermore equation (24) gives

$$(26) \quad p = p_0(\rho/\rho_0)^\gamma,$$

where p_0 and ρ_0 are the values of the pressure and density at $T=0$, i.e., on the shock surface and are functions of y_α .

If we put $\xi = \log(p)/\gamma$ then it follows from (26) that

$$(27) \quad \frac{1}{\rho} \frac{\partial \rho}{\partial T} = \frac{1}{\gamma p} \frac{\partial p}{\partial T} = \frac{\partial \xi}{\partial T}.$$

With the help of the relations (27) and (19), the equations (25) and (23) can respectively be written as

$$(28) \quad u_n \frac{\partial \xi}{\partial T} + N_i \frac{\partial u_i}{\partial T} = - \frac{\epsilon^{\alpha\beta} e_{ijk} u_i A_{k\beta}}{v} \frac{\partial u_j}{\partial y^\alpha},$$

and

$$(29) \quad c^2 N_i \frac{\partial \xi}{\partial T} + u_n \frac{\partial u_i}{\partial T} = - \frac{c^2 \epsilon^{\alpha\beta} e_{ijk} u_i A_{k\beta}}{2v} \frac{\partial \xi}{\partial y^\alpha},$$

where $c^2 = \gamma p / \rho$.

Now if an analytic shock surface is given and if flow incident upon it is also given, i.e., if we are given $a_{i\alpha}$, u_i , p_1 and ρ_1 , then the Cauchy-Kowaleski existence theorem for equations (28) and (29) enables us to state that there exists a unique analytic flow in a region behind the shock front provided that

$$(30) \quad v^4 \det \begin{vmatrix} u_n & N_1 & N_2 & N_3 \\ c^2 N_1 & u_n & 0 & 0 \\ c^2 N_2 & 0 & u_n & 0 \\ c^2 N_3 & 0 & 0 & u_n \end{vmatrix} = v^4 u_n^4 \left(1 - \frac{c^2 N^2}{u_n^2} \right) \neq 0,$$

i.e.,

$$(31) \quad 1 - \frac{c^2 N^2}{u_n^2} \neq 0.$$

If we make use of the shock relations (4) and the fact that $N^2 = n^2 = 1$, at the shock front, we readily obtain

$$(32) \quad 1 - \frac{c^2 N^2}{u_n^2} = - \frac{\gamma + 1}{2} \delta.$$

Since $\delta > 0$ for a shock of nonzero strength, the required condition is fulfilled. Thus there exists a unique analytic flow in a region behind a steady shock wave.

PART II. PSEUDO-STATIONARY FLOWS

In this case the three-dimensional shock configuration at time t may be represented by the equations [3]:

$$(33) \quad x_i = t a_i(y_1, y_2),$$

where y_1, y_2 are the Gaussian coordinates on the shock surface. The equations describing the motion of a fluid for a pseudo-stationary flow are then given as [3; 6]:

$$(34) \quad \rho U_{i,i} + \rho_{,i} U_i + 3\rho = 0, \text{ (equation of continuity),}$$

$$(35) \quad \rho(U_{i,j} U_j + U_i) + p_{,i} = 0, \text{ (equations of motion),}$$

$$(36) \quad S_{,i} U_i = 0, \text{ (the changes of state are adiabatic),}$$

where $U_i = u_i - a_i$. The shock conditions (4) remain the same except that u_i is changed to U_i in them [3]. In these equations comma denotes differentiation with respect to a_i .

If T represents the arc length along the streak line, a line having U_i/v as the components of its unit tangent vector, measured from the shock front [4], then we define, as in Part I, the transformation from a_i to y_1, y_2 and T such that a_i are the solutions of the equations

$$(37) \quad \frac{\partial a_i}{\partial T} = \frac{U_i}{v}; \quad v^2 = U_i U_i,$$

with the initial condition: $a_i(y_1, y_2, 0) = a_i(y_1, y_2)$.

The transformation from the variables a_i to the variables y_1, y_2, T thus completely defined, we proceed in a manner exactly similar to the one in Part I. The conservation equations (34) to (36) are transformed respectively into

$$(38) \quad \frac{U_n}{\rho} \frac{\partial \rho}{\partial T} + N_i \frac{\partial U_i}{\partial T} + \frac{3}{v} = - \frac{\epsilon^{\alpha\beta} e_{ijk} U_i A_{k\beta}}{v} \frac{\partial U_j}{\partial y^\alpha},$$

$$(39) \quad \frac{\partial U_i}{\partial T} + \frac{U_i}{v} + \frac{N_i}{\rho U_n} \frac{\partial p}{\partial T} = - \frac{\epsilon^{\alpha\beta} e_{ijk} U_j A_{k\beta}}{2\rho v} \frac{\partial p}{\partial y^\alpha},$$

$$(40) \quad p = p_0(\rho/\rho_0)^\gamma.$$

As in the stationary case if we put $\xi = \gamma^{-1} \log p$ and $c^2 = \gamma p/\rho$ then these equations reduce to

$$(41) \quad U_n \frac{\partial \xi}{\partial T} + N_i \frac{\partial U_i}{\partial T} = - \frac{\epsilon^{\alpha\beta} e_{ijk} U_i A_{k\beta}}{v} \frac{\partial U_j}{\partial y^\alpha} - \frac{3}{v},$$

and

$$(42) \quad c^2 N_i \frac{\partial \xi}{\partial T} + U_n \frac{\partial U_i}{\partial T} = - \frac{c^2 \epsilon^{\alpha\beta} e_{ijk} U_j A_{k\beta}}{v} \frac{\partial \xi}{\partial y^\alpha} - \frac{U_i U_n}{v}.$$

Comparing the set of equations (41) and (42) with the set (28) and (29) we see that the condition for the applicability of the Cauchy-Kowaleski existence theorem is the same in this case as in the case of stationary gas flows and is, therefore, satisfied. Hence, there exists a unique analytic flow behind a pseudo-stationary shock.

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NOTE ON A PAPER OF CIVIN AND YOOD

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It is possible to obtain stronger results than those of [1] in a small amount of space through the use of a different method. We follow the notation of [1] throughout.

THEOREM 1. *Let B be a complex commutative Banach algebra with countable space \mathfrak{M} of maximal regular ideals. Let A be a separating subalgebra. Then (i) if A is not contained in any maximal regular ideal it is determining. Otherwise (ii) there exists a (unique) maximal regular ideal M of B with \hat{A} dense in \hat{M} .*

PROOF. If $x \in A$ then $\text{Cl}[\hat{x}(\mathfrak{M})] = \hat{x}(\mathfrak{M}) \cup \{0\}$ is a countable compact set. Then by a theorem of Lavrent'ev [2], polynomials in z are uniformly dense on $\text{Cl}[\hat{x}(\mathfrak{M})]$ in all continuous complex valued functions on $\text{Cl}[\hat{x}(\mathfrak{M})]$ and in particular uniformly approximate \bar{z} on $\text{Cl}[\hat{x}(\mathfrak{M})]$, i.e. polynomials in \hat{x} uniformly approximate \hat{x}^- on \mathfrak{M} . Then, $\text{Cl}[\hat{A}]$ being a uniformly closed, separating, self adjoint algebra of continuous, complex valued functions vanishing at ∞ on a locally compact space \mathfrak{M} , equals $\mathcal{C}(\mathfrak{M})$ in case (i) or at worst some unique $\text{Cl}[\hat{\mathfrak{M}}]$ in case (ii) by the Stone-Weierstrass approximation theorem [3].

Theorem 3.3 and extensions of Theorem 3.5 and Corollary 3.6 of [1] eliminating the hypothesis of regularity and the necessity of limiting the discussion to maximal subalgebras follow as simple corollaries, e.g.

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