FACTORING A LATTICE\(^1\)
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For the purpose of this note a **lattice** is a Hasdorff space \(L\) together with a pair \(\wedge, \vee: L \times L \to L\) of continuous functions satisfying the usual conditions [3, p. 18]. The closed unit interval \(I\) with the operations \(\min, \max\) is a simple example. An **isomorphism** is a simultaneous isomorphism and homeomorphism (A. H. Clifford). It is known and easy to prove [1] that a closed and bounded interval admits a unique (modulo an isomorphism or a dual isomorphism) structure. It is known [1] that a 2-cell admits more than one lattice structure. We shall show that if no maximal chain cuts \(L\) into more than two pieces then the lattice structure is unique.

If \(L_1\) and \(L_2\) are lattices then \(L_1 \times L_2\) is the cartesian product of \(L_1\) and \(L_2\) with coordinate operations.

**Theorem.** If the lattice \(L\) is homeomorphic with a closed 2-cell and if no maximal chain cuts \(L\) into more than two components then \(L\) is isomorphic with \(I \times I\).

The proof is made by a series of lettered assertions.

(A) \(L\) has a zero and unit contained in its boundary [4].

Let \(S\) and \(T\) be the two arcs of the boundary of \(L\) with endpoints 0 and 1.

(B) The arcs \(S\) and \(T\) are maximal chains.

**Proof.** Here and later it is suggested that a simple figure will enhance the geometric flavor of the argument. Let \(x, y \in T\) with \(0 < x, y < 1\), let \(x\) precede \(y\) on \(T\) in the order from 0 to 1 and assume that \(x \in L \setminus y \wedge L\), i.e., it is false that \(x \leq y\). Then \(y \wedge L\) is compact and connected (continuous image of \(L\)), 0, \(y \in y \wedge L\) and \(x, 1 \in L \setminus y \wedge L\). Thus \(y \wedge L\) separates \(x\) and 1 on the boundary of \(L\) and thus (as is well-known, [5]) \(y \wedge L\) separates \(L\). Hence \(L \setminus y \wedge L = U \cup V\) where \(U\) and \(V\) are disjoint open sets with \(1 \in V\). If \(a \in U\) then \(a \vee L\) is a connected set containing \(a\) and 1 so that \((y \wedge L) \cap (a \vee L) \neq \emptyset\). But if \(b\) is in this set then \(a \leq b \leq y\) so that \(a \in y \wedge L\), a contradiction. Hence \(T\) is a chain, i.e., \(x, y \in T\) gives \(x \leq y\) or \(y \leq x\). Now let \(C\) be a maximal chain containing \(T\). Since \(L\) is compact and connected it is easy to see that \(C\) contains no proper connected subset containing 0 and 1. Indeed, \(C\) is an arc from 0 to 1 [1]. Hence \(T = C\) and the proof is complete.

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(C) The arcs $S$ and $T$ generate $L$ in the sense that $L = S \wedge T = S \vee T$.

Proof. Since $1 \in S \cap T$ we have $S \cup T \subseteq S \cap T$ and $T \cap (S \cup T) \subseteq T \cap S$. If $L \neq T \cap S$ let $f$ be a function retracting $T \cap S$ onto $T \cup S$, and define $g : T \times (T \cup S) \to T \cap S$ by $g(t, x) = t \wedge x$. Now $fg(0, x) = f(0) = 0$ and $fg(1, x) = f(x) = x$ since $x \in T \cup S$. The function $fg$ is then a homotopy between the identity and the constant map on $T \cup S$. This involves a contradiction.

(D) There are elements $b \in T \setminus \{0, 1\}$ and $b' \in S \setminus \{0, 1\}$ such that $b \wedge b' = 0$ and $b \vee b' = 1$.

Proof. Let

$$A = \{ x \mid x \wedge L \subseteq T \} \quad \text{and} \quad B = \{ x \mid x \vee L \subseteq T \}$$

and note that $A$ and $B$ are closed and $x \in A$ gives $x \in x \wedge L \subseteq T$ so that $A \cup B \subseteq T$. If $a$ is the least upper bound of $A$ and if $b$ is the greatest lower bound of $B$ we show that $b \leq a$. This being false let $p \in T$ such that $a < p < b$ (see (B)). Then $p \wedge L$ is not a part of $T$ so let $a_0 \in p \wedge L$. Then $a_0 = t \lor a_1$ with $t \in T$ and $a_1 \in S$. If $a_1 = 0$ then $a_0 \in T$. Thus $0 < a_1 < p$ since $a_1 \leq p$ and $a_1 = p$ gives $a_0 \in T \lor T \subseteq T$, by (B). In a dual fashion we show that $p < b_1 < 1$ for some $b_1 \in S$. Let $C$ be a maximal chain containing $0, a_1, p, b_1$ and $1$ so that $C$ is an arc in $L$ from $0$ to $1$ [1]. A figure will indicate that $L \setminus C$ has more than two components, a fact that can be verified by applying known results on $\theta$-curves [5, Chap. VI]. It can also be proved in consequence of the Jordan Theorem and the fact that a spanning arc of a 2-cell cuts the 2-cell, loc. cit. Since this contradicts our hypotheses we infer that $b \leq a$.

If $b = 1$ then $a = 1$ so that $L = a \wedge L \subseteq T$, an impossibility. Were $b = 0$ then $L = b \lor L \subseteq T$, so that $0 < b < 1$. Now $x \leq y \in A$ gives $x \in A$ so that $b \in A$ and hence $(b \wedge L) \cup (b \lor L) \subseteq T$. But $0, 1 \in (b \wedge L) \cup (b \lor L)$ and this set being connected we have $(b \wedge L) \cup (b \lor L) = T$. Dually, we locate $b' \in S$ with $0 < b' < 1$ and $(b' \wedge L) \cup (b' \lor L) = T$. From $S \cap T = \{0, 1\}$ an easy computation leads to $b \wedge b' = 0$ and $b \lor b' = 1$ and (D) is proved.

(E) The function $f : L \to (b \wedge L) \times (b' \lor L)$ defined by $f(x) = (b \wedge x, b' \lor x)$ is an isomorphism.

Proof. By a result of L. W. Anderson, we know that $L$ is distributive [2]. The arguments in §9, Chap. II of [3] and the fact that $L$ is compact complete the proof.

The proof of the theorem is finished if we note that $b \wedge L$ and $b' \wedge L$ are chains and hence arcs and so have the structure of $I$.

The content of the theorem is really the assertion that $L$ has a nontrivial center [3, p. 27] and (as pointed out to me by R. D. Ander-
son) it may be observed that the center of \( L \) has cardinal four. It has been conjectured that the center of any compact connected \( n \)-dimensional lattice has cardinal at most \( 2^n \).

L. W. Anderson ([1], to appear in Proc. Amer. Math. Soc.) has shown that a compact connected 1-dimensional lattice is a chain. It is an easy conjecture that a compact connected 2-dimensional metrizable lattice without cutpoints is a 2-cell. If it can be imbedded in the plane then it is a 2-cell. A proof of this fact (which is quite easy) will appear in the Pacific Journal of Mathematics. Thus, in the theorem of this note, the hypothesis that \( L \) be a 2-cell could be replaced by the assumptions that \( L \) is compact, connected, imbeddable in the plane and without cutpoints.

It is noted that (A), (B), and (C) are valid without the maximal chain condition on \( L \). In fact, these results can be reformulated so as to be valid for semigroups.

A. L. Shields obtained a result either identical with, or closely related to, a suitable reformulation of (C).

**Bibliography**


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