PREFERRED OPTIMAL STRATEGIES

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Let Γ be the normalized two person zero sum game defined by a pay-off function $M(x, y)$, for $x \in A$, $y \in B$. If A and B are compact convex sets in a finite dimensional space, and M is bilinear, then Γ is strictly determined. Then Γ has a value $v(\Gamma)$, and the players have optimal strategy sets $A_1 \subset A$, $B_1 \subset B$, such that $M(x, y_1) \leqq v(\Gamma) \leqq M(x_1, y)$ for any choices of $x \in A$, $x_1 \in A_1$, $y \in B$, $y_1 \in B_1$. We may denote the game Γ by $\langle M, A, B \rangle$. (See [**1**; **2**].)

This note is concerned with games in which the first player P_1 has more than one optimal strategy. Since A_1 is convex, there are then an infinite number. Against an optimal strategy of P_2, none of these will achieve more than $v(\Gamma)$. However, if P_2 should play nonoptimally, P_1 might obtain more than $v(\Gamma)$, and the outcome might depend upon which optimal strategy from the set A_1 he chooses. In many applications of game theory, it is desirable to have a systematic procedure for choosing a *preferred* strategy $\bar{x}$ in A_1 which will take advantage of the possibility of error (nonintelligent action) on the part of the second player. Such a procedure will be given in this note; the resulting preferred optimal strategy is unique, up to equivalence, when the set B is a polyhedron.

Two first player strategies, x' and x'', are said to be equivalent for the same Γ if $M(x', y) = M(x'', y)$ for all $y \in B$. When B is polyhedral, it has only a finite number of extreme points π. These we call "pure" strategies for P_2. Any $y \in B$ is then a finite convex combination of pure strategies. We divide the pure strategies of the second player into two classes. A pure strategy π is *good* if it is present in at least one optimal P_2 strategy (i.e. if it occurs with nonzero coefficient in an optimal strategy). All other pure strategies are called *poor*. The dichotomy can also be made analytically. If π is a good pure strategy, then $M(x_1, \pi) = v(\Gamma)$ for every optimal $x_1 \in A_1$; if π is a poor pure strategy, then there is at least one $x_1 \in A_1$ with $M(x_1, \pi) > v(\Gamma)$. Let $B^{(1)}$ be the closed convex hull of the set of poor pure strategies of P_2.

THEOREM 1. *The following statements are equivalent*: (i) $B^{(1)}$ *is void*, (ii) B_1 *contains a point interior to* B, (iii) *all the strategies in* A_1 *are equivalent.*

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Let B_* be the set of all points $y \in B$ such that $M(x, y) = v(\Gamma)$ for every optimal $x \in A_1$. This set is convex and contains B_1, but is disjoint from $B^{(1)}$. Moreover, any line segment in B which contains a point of B_* in its interior, lies wholly in B_*; thus, B_* is the convex hull of the set of good pure strategies. When (ii) holds, $B = B_*$. When (ii) fails, B_* is the face of B containing B_1, and $B \neq B_*$. Finally, it is immediate that (i) and (iii) are each equivalent to $B = B_*$.

Construct a new game, $\Gamma_1 = \langle M, A_1, B^{(1)} \rangle$. To this, we may apply the same procedure, generating a sequence of games $\Gamma_1, \Gamma_2, \Gamma_3 \cdots$ with $\Gamma_n = \langle M, A_n, B^{(n)} \rangle$. We have $A \supset A_1 \supset A_2 \supset \cdots$ and $B \supset B^{(1)} \supset B^{(2)} \supset \cdots$; A_{n+1} is the set of optimal strategies for P_1 in the game Γ_n, and $B^{(n+1)}$ is the convex hull of the poor pure strategies for P_2 in Γ_n. Moreover, the vertices of $B^{(n+1)}$ form a proper subset of those of $B^{(n)}$. When B is polyhedral, having a finite number of vertices, we must reach an integer N such that $B^{(N+1)}$ is void. In the game Γ_N, P_2 will have *no* poor pure strategies. By Theorem 1, then, all of the optimal strategies $\bar{x}$ in the set $A_\infty = A_{N+1}$ are then equivalent in the game Γ_N.

THEOREM 2. *The strategies $\bar{x}$ in A_∞ are all equivalent in* Γ.

Let $\bar{x} \in A_\infty$. Then, $\bar{x} \in A_n$ for any n. If π is any extreme point of B which is a good strategy for P_2 in Γ_n, then $M(\bar{x}, \pi) = v(\Gamma_n)$. Every extreme point π is good in Γ, or in one of the games Γ_j. Thus, $M(\bar{x}, \pi) = M(x, \pi)$, for every π and any choice of $x \in A_\infty$. Since B is the convex hull of the points π, $M(\bar{x}, y) = M(x, y)$ for every $y \in B$, and any choice of $x \in A_\infty$. Thus, all of the points of the set A_∞ are equivalent in Γ.

By this process, then, we have arrived at a strategy $\bar{x}$ which is optimal in *each* of the games $\Gamma, \Gamma_1, \Gamma_2, \cdots$, and which (when B is polyhedral) is unique, up to equivalence. When B is not polyhedral, the sequence $B^{(n)}$ may not terminate. However, the strategies in the set A_∞ still have the desirable properties described above, and are preferred optimal strategies.

We give a simple illustration. Consider the rectangular game whose (discrete) pay off matrix is

$$W = \begin{bmatrix} 6 & 1 & 5 & 8 & 5 \\ 2 & 7 & 3 & 4 & 4 \\ 6 & 1 & 7 & 4 & 5 \end{bmatrix}.$$

If Γ is the mixed game derived from W, so that A is a triangle and B a 4-simplex, then $v(\Gamma) = 4$, P_1 has two basic (extreme) optimal strategies $x' = (1/2, 1/2, 0)$, $x'' = (0, 1/2, 1/2)$ and P_2 has a unique optimal strategy $y=(3/5,\ 2/5,\ 0,\ 0,\ 0)$. The poor pure strategies for P_2 are columns 3, 4 and 5.

Proceeding as above, the game Γ_1 is then the mixed game obtained from the rectangular matrix

$$W = \begin{bmatrix} 4 & 6 & 9/2 \\ 5 & 4 & 9/2 \end{bmatrix}.$$

This was obtained by computing $[x',\ x'']W$, and deleting from this matrix the first two columns. The value of Γ_1 is $v(\Gamma_1)=9/2$, and P_1 has the optimal strategies (1/2, 1/2) and (1/4, 3/4). The last column is optimal for P_2. Repeating the process, Γ_2 is the mixed game obtained from

$$W = \begin{bmatrix} 9/2 & 5 \\ 19/2 & 9/2 \end{bmatrix}.$$

We see that $v(\Gamma_2)=14/3$, that P_1 has a unique optimal strategy (1/3, 2/3), and that $B^{(2)}$ is empty. Retracing our steps, we arrive at $\bar{x}=(1/6,\ 1/2,\ 1/3)$ which is the unique preferred optimal strategy. Note that its pay-off is $\bar{x}W=(4,\ 4,\ 14/3,\ 14/3,\ 9/2)$.

References

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