PREFERRED OPTIMAL STRATEGIES

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Let $\Gamma$ be the normalized two person zero sum game defined by a pay-off function $M(x, y)$, for $x \in A$, $y \in B$. If $A$ and $B$ are compact convex sets in a finite dimensional space, and $M$ is bilinear, then $\Gamma$ is strictly determined. Then $\Gamma$ has a value $v(\Gamma)$, and the players have optimal strategy sets $A_1 \subset A$, $B_1 \subset B$, such that $M(x, y_1) \leq v(\Gamma) \leq M(x_1, y)$ for any choices of $x \in A$, $x_1 \in A_1$, $y \in B$, $y_1 \in B_1$. We may denote the game $\Gamma$ by $(M, A, B)$. (See [1; 2].)

This note is concerned with games in which the first player $P_1$ has more than one optimal strategy. Since $A_1$ is convex, there are then an infinite number. Against an optimal strategy of $P_2$, none of these will achieve more than $v(\Gamma)$. However, if $P_2$ should play nonoptimally, $P_1$ might obtain more than $v(\Gamma)$, and the outcome might depend upon which optimal strategy from the set $A_1$ he chooses. In many applications of game theory, it is desirable to have a systematic procedure for choosing a preferred strategy $\hat{x}$ in $A_1$ which will take advantage of the possibility of error (nonintelligent action) on the part of the second player. Such a procedure will be given in this note; the resulting preferred optimal strategy is unique, up to equivalence, when the set $B$ is a polyhedron.

Two first player strategies, $x'$ and $x''$, are said to be equivalent for the same $\Gamma$ if $M(x', y) = M(x'', y)$ for all $y \in B$. When $B$ is polyhedral, it has only a finite number of extreme points $\pi$. These we call "pure" strategies for $P_2$. Any $y \in B$ is then a finite convex combination of pure strategies. We divide the pure strategies of the second player into two classes. A pure strategy $\pi$ is good if it is present in at least one optimal $P_2$ strategy (i.e. if it occurs with nonzero coefficient in an optimal strategy). All other pure strategies are called poor. The dichotomy can also be made analytically. If $\pi$ is a good pure strategy, then $M(x_1, \pi) = v(\Gamma)$ for every optimal $x_1 \in A_1$; if $\pi$ is a poor pure strategy, then there is at least one $x_1 \in A_1$ with $M(x_1, \pi) > v(\Gamma)$.

Let $B(1)$ be the closed convex hull of the set of poor pure strategies of $P_2$.

**Theorem 1.** The following statements are equivalent: (i) $B(1)$ is void, (ii) $B_1$ contains a point interior to $B$, (iii) all the strategies in $A_1$ are equivalent.

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Let $B_*$ be the set of all points $y \in B$ such that $M(x, y) = v(T)$ for every optimal $x \in A_1$. This set is convex and contains $B_1$, but is disjoint from $B^{(1)}$. Moreover, any line segment in $B$ which contains a point of $B_*$ in its interior, lies wholly in $B_*$; thus, $B_*$ is the convex hull of the set of good pure strategies. When (ii) holds, $B = B_*$. When (ii) fails, $B_*$ is the face of $B$ containing $B_1$, and $B \neq B_*$. Finally, it is immediate that (i) and (iii) are each equivalent to $B = B_*$.  

Construct a new game, $\Gamma = \langle M, A_1, B^{(1)} \rangle$. To this, we may apply the same procedure, generating a sequence of games $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ with $\Gamma_n = \langle M, A_n, B_n^{(n)} \rangle$. We have $A \supset A_1 \supset A_2 \supset \cdots$ and $B \supset B^{(1)} \supset B^{(2)} \supset \cdots$; $A_{n+1}$ is the set of optimal strategies for $P_1$ in the game $\Gamma_n$, and $B_{n+1}$ is the convex hull of the poor pure strategies for $P_2$ in $\Gamma_n$. Moreover, the vertices of $B^{(n+1)}$ form a proper subset of those of $B^{(n)}$. When $B$ is polyhedral, having a finite number of vertices, we must reach an integer $N$ such that $B^{(N+1)}$ is void. In the game $\Gamma_N$, $P_2$ will have no poor pure strategies. By Theorem 1, then, all of the optimal strategies $\hat{x}$ in the set $A_\infty = A_{N+1}$ are then equivalent in the game $\Gamma_N$.

**Theorem 2.** The strategies $\hat{x}$ in $A_\infty$ are all equivalent in $\Gamma$.

Let $\hat{x} \in A_\infty$. Then, $\hat{x} \in A_n$ for any $n$. If $\pi$ is any extreme point of $B$ which is a good strategy for $P_2$ in $\Gamma_n$, then $M(\hat{x}, \pi) = v(\Gamma_n)$. Every extreme point $\pi$ is good in $\Gamma$, or in one of the games $\Gamma_j$. Thus, $M(\hat{x}, \pi) = M(x, \pi)$, for every $\pi$ and any choice of $x \in A_\infty$. Since $B$ is the convex hull of the points $\pi$, $M(\hat{x}, y) = M(x, y)$ for every $y \in B$, and any choice of $x \in A_\infty$. Thus, all of the points of the set $A_\infty$ are equivalent in $\Gamma$.

By this process, then, we have arrived at a strategy $\hat{x}$ which is optimal in each of the games $\Gamma, \Gamma_1, \Gamma_2, \ldots$, and which (when $B$ is polyhedral) is unique, up to equivalence. When $B$ is not polyhedral, the sequence $B^{(n)}$ may not terminate. However, the strategies in the set $A_\infty$ still have the desirable properties described above, and are preferred optimal strategies.

We give a simple illustration. Consider the rectangular game whose (discrete) pay off matrix is:

$$W = \begin{bmatrix} 6 & 1 & 5 & 8 & 5 \\ 2 & 7 & 3 & 4 & 4 \\ 6 & 1 & 7 & 4 & 5 \end{bmatrix}.$$  

If $\Gamma$ is the mixed game derived from $W$, so that $A$ is a triangle and $B$ a 4-simplex, then $v(\Gamma) = 4$, $P_1$ has two basic (extreme) optimal strategies $x' = (1/2, 1/2, 0)$, $x'' = (0, 1/2, 1/2)$ and $P_2$ has a unique optimal
strategy \( y = (3/5, 2/5, 0, 0, 0) \). The poor pure strategies for \( P_2 \) are columns 3, 4 and 5.

Proceeding as above, the game \( \Gamma_1 \) is then the mixed game obtained from the rectangular matrix

\[
\begin{bmatrix}
4 & 6 & 9/2 \\
5 & 4 & 9/2 
\end{bmatrix}
\]

This was obtained by computing \([x', x'']^T W\), and deleting from this matrix the first two columns. The value of \( \Gamma_1 \) is \( v(\Gamma_1) = 9/2 \), and \( P_1 \) has the optimal strategies \((1/2, 1/2)\) and \((1/4, 3/4)\). The last column is optimal for \( P_2 \). Repeating the process, \( \Gamma_2 \) is the mixed game obtained from

\[
\begin{bmatrix}
9/2 & 5 \\
19/2 & 9/2 
\end{bmatrix}
\]

We see that \( v(\Gamma_2) = 14/3 \), that \( P_1 \) has a unique optimal strategy \((1/3, 2/3)\), and that \( B^{(2)} \) is empty. Retracing our steps, we arrive at \( \hat{x} = (1/6, 1/2, 1/3) \) which is the unique preferred optimal strategy. Note that its pay-off is \( \hat{x} W = (4, 4, 14/3, 14/3, 9/2) \).

References


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