THE CONVEX HULL OF SUB-PERMUTATION MATRICES

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1. Introduction. A combinatorial theorem [1; 3] usually referred to as "the marriage problem" or "the problem of distinct representatives" has the following matrix formulation; the convex hull of the set of all \( n \) by \( n \) permutation matrices is the set of all \( n \) by \( n \) doubly stochastic matrices. In this note the above theorem is generalized.

The following notation and definitions will be used. \( A \) will represent an \( n \) by \( n \) matrix with non-negative real entries \( a_{ij} \); \( S \) will represent the sum of all entries of \( A \), \( S = \sum_i \sum_j a_{ij} \); \( R_i \) will represent the sum of the entries in the \( i \)th row and \( C_j \) will represent the sum of the entries in the \( j \)th column; \( M \) will represent the largest row or column sum of \( A \), \( M = \max (R_i, C_j) \). Also used will be the concept of a sub-permutation matrix of rank \( r \). By this is meant a matrix \( P \) with the following properties: (1) each entry of \( P \) is either 1 or 0; (2) each row and each column of \( P \) contains at most one 1; (3) \( P \) contains exactly \( r \) entries equal to 1. In terms of this notation the theorem quoted above becomes; a matrix \( A \) lies in the convex hull of the set of all permutation matrices if and only if \( M = 1 \) and \( S = n \). In [2] the authors of the present note obtain sufficient conditions in order that a matrix \( A \) with non-negative entries contain nonzero entries in the places occupied by 1 in a permutation matrix of rank \( r \). In this note necessary and sufficient conditions are given in order that a matrix \( A \) lie in the convex hull of the sub-permutation matrices of rank \( n - i \) (\( i = 0, 1, 2, \ldots, n - 1 \)).

2. The Theorem. Let \( A \) be an \( n \) by \( n \) matrix whose entries are non-negative real numbers. A necessary and sufficient condition that \( A \) lie in the convex hull of all sub-permutation matrices of rank \( n - i \) is that \( S = n - i \) and \( (n - i)/n \leq M \leq 1 \).

Proof. The necessity is obtained as follows. Let \( A = \sum_j \alpha_j P_j \) where \( \alpha_j \geq 0 \), \( \sum_j \alpha_j = 1 \) and \( P_j \) is a sub-permutation matrix of rank \( n - i \). Then each matrix \( \alpha_j P_j \) has the sum of all its entries equal to \((n - i)\alpha_j \) and each row or column sum has the value \( \alpha_j \) or 0. Hence \( S = (n - i) \sum_j \alpha_j = (n - i) \) and \( M \leq \sum_j \alpha_j = 1 \). Also since \( n - i = S = \sum_j R_j \leq nM \), \((n - i)/n \leq M \). Hence \( S = n - i \) and \( (n - i)/n \leq M \leq 1 \).

To obtain the sufficiency we note that if \( S = n - i \) and \( (n - i)/n \leq M \leq 1 \).
\[ \sum R_j = \sum C_j = n - i. \]

Also the numbers \( 1 - R_1, 1 - R_2, \ldots, 1 - R_n \) are non-negative and at least one of these is positive if \( i > 0 \). For if all of \( 1 - R_1, 1 - R_2, \ldots, 1 - R_n \) were 0 then \( R_j = 1 = M \) for all \( j \) so that \( S = n \) a contradiction. The matrix \( A \) is now augmented to a matrix \( A^* \) by the addition of \( i \) rows and \( i \) columns as follows: \( a^*_{rs} = a_{rs} \) if \( r \) and \( s \) are both less than or equal to \( n \); \( a^*_{rs} = 0 \) if \( r \) and \( s \) are both greater than \( n \); \( a^*_{n+t, v} = (1 - R_v)/i \) for \( r = 1, 2, \ldots, n; t = 1, 2, \ldots, i; u = 1, 2, \ldots, i; v = 1, 2, \ldots, n. \) The matrix \( A^* \) is a doubly stochastic \( n+i \) by \( n+i \) matrix with zeros in the lower right hand \( i \) by \( i \) block. By the theorem quoted in the introduction \( A^* = \sum \alpha_r P^*_r \) where \( \alpha_r \geq 0, \sum \alpha_r = 1 \) and \( P^*_r \) is an \( n+i \) by \( n+i \) permutation matrix. Furthermore, each \( P^*_r \) has an \( i \) by \( i \) block of zeros in its lower right corner. Hence \( P^*_r \) has \( 2i \) entries equal to 1 in its last \( i \) rows and \( i \) columns. If \( P_r \) is the \( n \) by \( n \) matrix in the upper left hand corner of \( P^*_r, P_r \) contains \( (n+i) - 2i = n - i \) ones. Hence \( P_r \), is a sub-permutation matrix of rank \( n - i \). Also \( A = \sum \alpha_r P_r. \)

**Bibliography**


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