LIOUVILLE'S THEOREM FOR PARABOLIC EQUATIONS OF THE SECOND ORDER WITH CONSTANT COEFFICIENTS[1]

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1. We shall consider a generalization of Liouville's Theorem for functions which are solutions of the parabolic equation

$$\frac{\partial u(x, t)}{\partial t} = \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 u(x, t)}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i \frac{\partial u(x, t)}{\partial x_i} + cu(x, t). \tag{1}$$

The coefficients a_{ij}, b_i and c are real constants, (a_{ij}) is a positive matrix and the solutions $u(x, t)$ are defined and nonnegative in the half space $-\infty < t \leqq 0$, denoted by D. We use the notation $x = (x_1, \cdots, x_m)$, $|x| = (\sum x_i^2)^{1/2}$.

It will be shown later on, that the nontrivial solutions of (1) are positive in the interior of D and that

$$\lim_{t \to -\infty} \frac{\log u(0, t)}{t} \text{ exists.} \tag{2}$$

The growth properties of these solutions will be studied in cones D_α: $|x| \leqq \alpha |t|$, $t<0$ with axis $x=0$ and opening 2α. The natural generalization of Liouville's Theorem (see, for instance, [1]) is equivalent to the statement that the nonconstant solutions of (1) cannot be bounded in D, which may be considered as a cone D_α with $\alpha = \infty$. We can now state the main result of this paper.

THEOREM 1. *Let $u(x, t)$ be a nontrivial nonnegative solution of* (1) *in the half space D. Denote*

$$\lim_{t \to -\infty} \frac{\log u(0, t)}{t} = c + \gamma. \tag{3}$$

If $c+\gamma \geqq 0$ and if $\gamma > 0$, then $u(x, t)$ is unbounded in D_α for all

$$\alpha > (c + \gamma)\left(\left(\gamma + \frac{1}{4} b^2\right)^{1/2} - \frac{1}{2} b\right)^{-1}, \tag{4}$$

where $b = (\sum b_i^2)^{1/2}$.

The assumption $c+\gamma \geqq 0$ excludes the trivial case $c+\gamma<0$, in

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which $u(0, t)$ is unbounded. The assumption $\gamma > 0$ is sharp in the sense that if $\gamma = 0$, u may be bounded in D. As an example, take $u(x, t) = e^{ct}$ where $c > 0$; $u(x, t)$ satisfies $\partial u/\partial t = \Delta u + cu$, but it is bounded in D. Note finally, that the assumption $\gamma > 0$ excludes constant ($\neq 0$) solutions. The proof of Theorem 1 is given in the next two sections.

2. In this section we shall consider some growth properties of a special class of *nontrivial* solutions of the heat equation

$$\frac{\partial w(x, t)}{\partial t} = \Delta w(x, t), \tag{5}$$

in the half space D. These solutions are assumed to be nonnegative and symmetric in x with respect to the axis $x = 0$, i.e., $w(x, t) = u(|x|, t)$. Separating variables in (5), we easily find that

$$e^{\xi^2 t} K_m(|x| \xi) = e^{\xi^2 t} \sum_{k=0}^{\infty} \frac{(\xi |x|)^{2k}}{4^k k! \Gamma(k + m/2)} \tag{6}$$

is a nonnegative symmetric solution of (5), and by superposition, we obtain solutions of the form

$$w(x, t) = u(|x|, t) = \int_0^\infty e^{\xi^2 t} K_m(|x| \xi) d\phi(\xi) \qquad \left(\phi(\xi) \nearrow ; 0 < \int_0^\infty d\phi(\xi) < \infty\right). \tag{7}$$

Hirschman [2] has proved that every nonnegative symmetric solution of (5) can be written in the form (7). The representation (7) will play an essential role in the following.

We can define

$$h(\alpha) = \limsup_{t \to -\infty} \frac{\log u(\alpha |t|, t)}{t}, \tag{8}$$

since, as follows from (7), the nonnegative symmetric solutions of (5) in D are positive in the interior of D. Let α_0 denote the distance from the origin to the support of $d\phi(\xi)$.

LEMMA 1. *If* $0 \leqq \alpha \leqq \alpha_0$; *then* $h(\alpha) = \alpha_0(\alpha_0 - \alpha)$.

PROOF. Consider first the function

$$v(\alpha |t|, t) = \int_0^\infty e^{(\xi - \alpha)\xi t} d\phi(\xi). \tag{9}$$

From the definition of α_0 it follows that for every $\epsilon>0$

$$v(\alpha|t|, t) = \left(\int_{\alpha_0}^{\alpha_0+\epsilon} e^{\xi(\xi-\alpha)t} d\phi(\xi)\right)(1 + o(1)) \qquad (t \to -\infty).$$

Consequently,

$$A_1 e^{(\alpha_0+\epsilon)(\alpha_0+\epsilon-\alpha)t} \leqq v(\alpha|t|, t) \leqq A_2 e^{\alpha_0(\alpha_0-\alpha)t} \quad (A_1 > 0,\ A_2 > 0), \tag{10}$$

from which it follows that

$$\limsup_{t\to-\infty} \frac{\log v(\alpha|t|, t)}{t} = \alpha_0(\alpha_0 - \alpha). \tag{11}$$

Using the inequalities

$$A' K_m(\alpha|t|\xi) \leqq e^{\alpha\xi|t|} \leqq A K_m(\alpha(1+\epsilon)|t|\xi) \qquad (A' > 0,\ A = A(\epsilon) > 0), \tag{12}$$

which follow by comparing the corresponding power series expansions, and using the definitions (8) and (9), we conclude from (11) that

$$\alpha_0(\alpha_0 - \alpha) \leqq h(\alpha), \qquad h(\alpha(1+\epsilon)) \leqq \alpha_0(\alpha_0 - \alpha).$$

Replacing α in the second inequality by $\alpha/(1+\epsilon)$ and taking $\epsilon\to 0$, we obtain $h(\alpha) \leqq \alpha_0(\alpha_0-\alpha)$, which together with the first inequality proves the lemma.

From the inequalities (10) it is clear that we have actually proved that $\lim_{t\to-\infty} (\log u(\alpha|t|, t))/t$ exists and is equal to $\alpha_0(\alpha_0-\alpha)$. Taking in particular $\alpha=0$, we obtain

$$\alpha_0^2 = \lim_{t\to-\infty} \frac{\log u(0, t)}{t}. \tag{13}$$

LEMMA 2. *If* $\alpha_0<\alpha<\infty$, *then* $h(\alpha) \leqq \alpha_0(\alpha_0-\alpha)$.

PROOF. As in the proof of the preceding lemma, we first consider the functon $v(\alpha|t|, t)$ defined by (9).

Since $d\phi(\xi) \not\equiv 0$ immediately to the right of $\xi=\alpha_0$, we get

$$\int_{\alpha_0}^{\alpha} e^{\xi(\xi-\alpha)t} d\phi(\xi) \geqq A e^{(\alpha_0+\epsilon)(\alpha_0+\epsilon-\alpha)t} \qquad (\epsilon > 0,\ A = A(\epsilon) > 0),$$

from which it follows that

$$\limsup_{t\to-\infty} \frac{\log v(\alpha|t|, t)}{t} \leqq (\alpha_0+\epsilon)(\alpha_0+\epsilon-\alpha).$$

Taking $\epsilon \to 0$ and proceeding as in the proof of Lemma 1, we finally get $h(\alpha) \leqq \alpha_0(\alpha_0 - \alpha)$.

3. Let $z(x, t)$ be a nontrivial nonnegative solution of the heat equation in D. Define

$$u(r, t) = \frac{1}{\omega_m r^{m-1}} \int_{|x|=r} z(x, t) dS_x,$$

where ω_m is the surface area of the m-dimensional unit sphere. $u(|x|, t)$ is also a solution of the heat equation [2] and, consequently, belongs to the class of functions considered in the preceding section. Denoting

$$M_t^z(r) = \operatorname*{Max}_{|x|=r} z(x, t), \tag{14}$$

and applying Lemmas 1, 2 and (13), we obtain

LEMMA 3. *For every nonnegative* α,

$$\limsup_{t \to -\infty} \frac{\log M_t^z(\alpha |t|)}{t} \leqq \alpha_0(\alpha_0 - \alpha) \tag{15}$$

where α_0 *is given by*

$$\alpha_0^2 = \lim_{t \to -\infty} \frac{\log z(0, t)}{t} \cdot \tag{16}$$

With the aid of Lemma 3 we proceed to prove Theorem 1. Let $T = (t_{ij})$ be an orthogonal matrix such that the function $v(x, t) = u(Tx, t)$ satisfies the equation

$$\frac{\partial v(x, t)}{\partial t} = \Delta v(x, t) + \sum_{i=1}^{m} d_i \frac{\partial v(x, t)}{\partial x_i} + cv(x, t).$$

Clearly, $d = (\sum d_i^2)^{1/2} = (\sum b_i^2)^{1/2} = b$. The function $w(x, t) = e^{\Sigma d_i x_i/2} v(x,t)$ satisfies the equation

$$\frac{\partial w(x, t)}{\partial t} = \Delta w(x, t) + (c - b^2/4) w(x, t),$$

and finally, the function $z(x, t) = e^{-(c-b^2/4)t} w(x, t)$ is a nontrivial nonnegative solution of the heat equation. Since

$$u(Tx, t) = e^{(c-b^2/4)t} e^{-\Sigma d_i x_i/2} z(x, t), \tag{17}$$

it follows that $u(x, t)$ is positive in the interior of D and that

$$\lim_{t\to-\infty}\frac{\log u(0,t)}{t}=\left(c-\frac{1}{4}b^2\right)+\lim_{t\to-\infty}\frac{\log z(0,t)}{t}. \tag{18}$$

From (17) we also deduce that

$$\begin{aligned}\limsup_{t\to-\infty}&\frac{\log M_t^u(\alpha\,|t|)}{t}\\ &\leqq\left(c-\frac{1}{4}b^2\right)+\frac{1}{2}b\alpha+\limsup_{t\to-\infty}\frac{\log M_t^z(\alpha\,|t|)}{t}.\end{aligned} \tag{19}$$

From (3) and (18) we have

$$\lim_{t\to-\infty}\frac{\log z(0,t)}{t}=\gamma+\frac{1}{4}b^2,$$

and using Lemma 3 we obtain,

$$\limsup_{t\to-\infty}\frac{\log M_t^z(\alpha\,|t|)}{t}\leqq\nu+b^2/4-(\gamma+b^2/4)^{1/2}\alpha.$$

Substituting in (19) and taking $\alpha>(c+\gamma)((\gamma+b^2/4)^{1/2}-b/2)^{-1}$, we get

$$\limsup_{t\to-\infty}\frac{\log M_t^u(\alpha\,|t|)}{t}<0.$$

Since for bounded $M_t^u(\alpha|t|)$ $(-\infty<t\leqq 0)$

$$\limsup_{t\to-\infty}(\log M_t^u(\alpha\,|t|))/t\geqq 0,$$

the theorem is proved.

4. In the special case $b=0$, (4) becomes $\alpha>(c+\gamma)\gamma^{-1/2}$. In the following theorem we release the assumption $\gamma>0$.

THEOREM 2. *Let $u(x, t)$ be a nontrivial nonnegative solution of*

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{m}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+cu \tag{20}$$

in the half space D and let (a_{ij}) be a positive matrix. If $u(x,t)\not\equiv Ae^{ct}$, then $u(x,t)$ cannot be bounded in every cone D_α.

PROOF. From (19) it is clear that if α satisfies

$$h(\alpha)+c<0, \tag{21}$$

then $u(x, t)$ is unbounded in D_α. Here, $h(\alpha)$ is defined with respect to the function $v(|x|, t)$ obtained by symmetrization of $e^{-ct}u(Tx, t)$. Since the last function is not constant, $v(|x|, t)$ is also not constant, and a simple argument based on (7) shows that $h(\alpha)\to -\infty$ as $\alpha\to\infty$. It follows that (21) is satisfied for large α.

In the case $c=0$ we can solve (21) by using Lemmas 1 and 2. We get

COROLLARY. *Let $u(x, t)$ be a nonconstant solution in D of the equation*

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} .$$

Suppose further that $u(x, t)$ is bounded from below. Then $u(x, t)$ is unbounded in D_α for all

$$\alpha > \left(\lim_{t\to-\infty} \frac{\log u(0, t)}{t} \right)^{1/2} .$$

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