ON FREE PRODUCTS[1]

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We prove several theorems for free products most of which are generalizations of results known for free groups.

Let G be a nontrivial free product of finitely many groups A_i each of which is a finite extension of a free group F_i of finite rank $\geqq 0$. Then we call G a *free product of finite type*. In particular a free product of finitely many finite groups is of finite type. In what follows "f.i." denotes "finite index" and "f.g." denotes "finitely generated."

THEOREM 1. *Let G be a free product of finite type and $H \neq 1$, a f.g. subgroup of G. If H is normal (or more generally if H contains a nontrivial normal subgroup of G) then H is of f.i. in G.*

PROOF. Let F be the normal subgroup of G generated by the F_i and the commutators $[A_i, A_j]$ $(i \neq j)$. We show F is a free group. For, by the Kurosh Subgroup Theorem (see [6]), F is the free product of a free group and the intersections of F with various conjugates of the A_i. The intersection of F with a conjugate of A_i is a conjugate of $F \cap A_i$. It therefore suffices to show $F \cap A_i$ is free. Consider the natural homomorphism ν of G onto G/F. Now G/F is isomorphic to the direct product of the groups A_i/F_i. Moreover, under ν, A_i is mapped canonically onto A_i/F_i. Hence $F \cap A_i$, which is the kernel of ν restricted to A_i, must be F_i. Clearly then F is a free group of f.i. in G.

Let $N \neq 1$ be a normal subgroup of G contained in H. Then N must be infinite. For otherwise, applying the Kurosh Subgroup Theorem, we would have N contained in a conjugate of some A_i. But then the normalizer of N would also be in this conjugate (see proof of Theorem 5), contrary to N being normal in G. By the isomorphism theorem $N/N \cap F \simeq NF/F$ and $H/H \cap F \simeq HF/F$. Hence $N \cap F$ is of f.i. in N so that $N \cap F \neq 1$. Also $H \cap F$ is of f.i. in the f.g. group H, and so $H \cap F$ is f.g. (see Schreier [9]). Thus we have $H \cap F$ is a f.g. subgroup of a free group G containing $N \cap F$, a nontrivial normal subgroup of F. Consequently (see [5]), $H \cap F$ is of f.i. in F and so H is of f.i. in G.

COROLLARY 1. *Let G be a free product of finite type, and H a f.g.*

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subgroup. If H is of infinite index then the intersection of the conjugates of H must be 1.

COROLLARY 2. *If G is a free product of finite type, and H_1, H_2 are f.g. subgroups, then $H_1 \cap H_2$ is f.g.*

PROOF. First observe that *if G is any group, K a subgroup of f.i., H a f.g. subgroup, then $H \cap K$ is f.g.*, since $H \cap K$ is of f.i. in H.

Now let F be as in the proof of Theorem 1. Then $H_1 \cap F$, $H_2 \cap F$ are f.g. subgroups of the free group F and therefore $H_1 \cap H_2 \cap F$ is f.g. (see Howson [4]), and is of f.i. in $H_1 \cap H_2$. Hence a set of generators for $H_1 \cap H_2 \cap F$ together with coset representatives for $H_1 \cap H_2$ mod $H_1 \cap H_2 \cap F$ constitutes a finite set of generators for $H_1 \cap H_2$.

In particular, taking G to be the modular group, we see that the intersection of two f.g. groups of unimodular 2×2 integral matrices is f.g.

THEOREM 2. *If G is a free product of finite type, then G is hopfian, i.e. $G \simeq G/N$ implies $N=1$.*

PROOF. Let G_k denote the intersection of all subgroups of G of index $\leqq k$. Then $\bigcap_1^\infty G_k = 1$. For if F is as above, then the intersection of the subgroups of f.i. in F (and therefore in G) is 1 (see e.g. Kurosh [6]). Let $G \simeq G/N$ and let ν be the natural homomorphism of G onto G/N. Since G is f.g. it has only finitely many subgroups of f.i. $\leqq k$ (see Hall [2]) and the number of these is the same as the number of subgroups of f.i. $\leqq k$ in G/N. But the pre-image under ν of such a subgroup of G/N is a subgroup of G, of f.i. $\leqq k$, and containing N. Hence $G_k \supset N$ for each k, and so $N=1$.

COROLLARY 3. *If G is any f.g. group such that the intersection of its subgroups of f.i. is* 1, *then G is hopfian.*

COROLLARY 4. *Let G be any f.g. group and let N be the intersection of its subgroups of f.i. Then G/N is hopfian, and moreover $G/N \simeq G/K$ implies $N=K$.*

PROOF. Clearly the intersection of the subgroups of f.i. in G/N is 1, and so G/N is hopfian. Moreover, G/N and therefore G/K, have as many subgroups of f.i. m as G has. Hence, as in the proof above, each subgroup of f.i. in G contains K. Therefore $N \supset K$ so that $G/K \simeq G/N \simeq (G/K)/(N/K)$. But since G/N is hopfian, G/K is hopfian and so $N=K$.

That Theorem 2 does not hold for any f.g. nontrivial free product follows immediately from the existence of a f.g. non-hopfian group H (see Neumann [8] or Higman [3]). For, let $H \simeq H/K$, $K \neq 1$. If

$G=H*L$, L any f.g. group and N the normal subgroup of G generated by K, then $G/N \simeq (H/K)*L \simeq H*L=G$.

A group G is *ω-nilpotent* (*ω-solvable*) if the intersection of the groups of the lower central series (derived series) is the identity. Next a generalization of a theorem of W. Magnus.

THEOREM 3. *If G is a free product of finite type and each A_i is a prime power extension of F_i corresponding to the same prime, then G is ω-nilpotent. Moreover, if the A_i are finite, then G is ω-nilpotent if and only if the A_i are prime power groups corresponding to the same prime.*

PROOF. Let F be as above. If the A_i/F_i are p-groups, then G/F is a p-group. Form the groups $Q_0=F$, $Q_{j+1}=$ the intersection of all normal subgroups of index p in Q_j. We may now proceed as for free groups (see Hall [2]) to obtain $\bigcap Q_j=1$ and that the intersection of the groups of the lower central series of G is 1.

Now, if some A_i has an element of order p, and some A_j $(j\neq i)$ has an element of order q, p, q distinct primes, then G has a subgroup which is the free product of a cyclic group of order p by a cyclic group of order q. This group is not ω-nilpotent (see Takahasi [10]). Hence G cannot be ω-nilpotent. In particular if the A_i are finite, they must all be p-groups.

THEOREM 4. *If G is a free product of finite type and each A_i/F_i is solvable, then G is ω-solvable. Moreover, if the A_i are finite, then G is ω-solvable if and only if the A_i are solvable.*

PROOF. Let F be as above. Then G/F is solvable and F is ω-solvable (since it is free). Hence the kth group $G^{(k)}$ of the derived series of G is in F so that $G^{(k+n)}\subset F^{(n)}$. Thus G is ω-solvable. Moreover, if G is ω-solvable, each subgroup is ω-solvable, and therefore A_i finite implies A_i solvable.

THEOREM 5. *Let G be an arbitrary nontrivial free product and $H^{(\neq 1)}$ a subgroup having finitely many conjugates H_i. Then $\bigcap H_i\neq 1$.*

PROOF. Let A be one of the factors of G, and let $a\in A$, $a\neq 1$, $v\in G$. Then $v^{-1}av\in A$ implies $v\in A$. For, let $v=b_1b_2\cdots b_r$ be the reduced form of v in the free product G. Then $v^{-1}av$ will have syllable length >1 unless $b_1\in A$. In that case $b_1^{-1}ab_1\in A$, and its conjugate by $b_2\cdots b_r$ is in A. Therefore, by induction, $b_2\cdots b_r$ and so v must be in A. Consequently, the normalizer of a subset of a conjugate of A must be in that conjugate of A.

Moreover, if h is not in a conjugate C_j of a factor of G, then the normalizer N_h of h is infinite cyclic. For, the preceding remark implies that N_h is disjoint from the C_j, which in turn implies (by the Kurosh Subgroup Theorem) that N_h is a free group. Since every element of N_h commutes with h, h must be in each cyclic free factor of N_h. Therefore, N_h is infinite cyclic.

Next, suppose H, K ($\neq 1$) are normal subgroups of S, a subgroup of G not contained in any C_j. Then $H \cap K \neq 1$. For otherwise, each element of H commutes with each element of K. If H intersects some C_j in more than the identity, then $K \subset C_j$ and therefore $S \subset C_j$. Hence $\exists h \in H$, with h in no C_j, and $K \subset N_h$. Then K and the cyclic group generated by h intersect nontrivially, since N_h is infinite cyclic. Thus $H \cap K \neq 1$.

Finally, let $H_1, \cdots, H_n$ be the conjugates of H. Now N_{H_i} (the normalizer of H_i) is of f.i. in G and so $N = \bigcap_1^n N_{H_i}$ is of f.i. in G. Since H_i is infinite, $N \cap H_i \neq 1$. But since N is in no C_j (C_j cannot be of f.i. in G) we can apply the remark of the preceding paragraph to obtain $\bigcap_1^n H_i = \bigcap^n (N \cap H_i) \neq 1$.

COROLLARY 5. *Let G be a free product of finite type and let H be a f.g. subgroup. Then the following three conditions are equivalent*:

(a) *H is of f.i.*,

(b) *H has finitely many conjugates, and*

(c) *H contains the dth powers of all elements of G for some* $d > 0$.

PROOF. That (a) and (b) are equivalent follows from Theorems 1 and 5. For the equivalence of (a) and (c) see [5].

COROLLARY 6. *Let G be a free product of cyclic groups. If two elements of G commute then they must be powers of the same element.*

In particular, taking G to be the modular group, it follows that if two unimodular 2×2 integral matrices commute then they must be powers of the same unimodular matrix. This result was first proved by K. Goldberg [1] and later generalized by O. Taussky and J. Todd [11] to matrices over a complex quadratic field. The methods used in both papers are matrix methods.

That a non-normal subgroup H of a free product can be of infinite index and yet have only finitely many conjugates can be seen as follows: Let F be the free group on a, b, and let K be the subgroup of index 2 in F consisting of all words of even length. It can be shown that K is freely generated by a^2, ab, ba. Let H be the normal subgroup of K generated by ab. Clearly $K:H$, and hence $F:H$ is infinite. Now $a^{-1}(ab)a = ba$ is not in H. Therefore the normalizer of H in F is just K. Thus H is of infinite index in F and yet has only two conjugates.

Let G be an extension of a f.g. free group F by a finite group. Using the same methods we could conclude: G is hopfian; G/F is a p-group implies G is ω-nilpotent; G/F is solvable implies G is ω-solvable. Theorems 1 and 5 hold for G if we assume the subgroup H is infinite. This last remark implies that the Braid Group can not be a finite extension of a free group, since it contains a f.g. infinite normal subgroup of infinite index, namely the normal divisor N, the quotient group of which is the mapping-class group of a sphere with n holes, n being the degree of the Braid Group. (See Magnus [7].)

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