ON THE COMPLEX FORM OF THE POINCARÉ LEMMA[1]

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1. **Introduction.** On a real differentiable (i.e. C^∞) manifold M of dimension n, any differential form can be decomposed into a sum of forms of degree r, $0 \leqq r \leqq n$, and the operator d of exterior differentiation ($d^2=0$) is an operator of degree $+1$. A differential form ϕ satisfying $d\phi=0$ is called closed, and a form ϕ such that $\phi=d\psi$ is called exact. The Poincaré Lemma states that a closed differential form of positive degree is locally exact; that is, if ϕ is defined and satisfies $d\phi=0$ in an open $U\subset M$, and m_0 is any point of U, then there exists an open neighborhood V of m_0, with $V\subset U$, and a form ψ defined in V such that $\phi=d\psi$ in V. This result is most conveniently proved by choosing a V differentiably contractible to the point m_0, for which there can be constructed an operator k of degree -1 satisfying

$$dk\phi + kd\phi = \phi \tag{1.1}$$

for all forms on V of positive degree, and then setting $\psi=k\phi$ (e.g. [**3**]).

In the case of a complex analytic manifold M of complex dimension n (real dimension $2n$), the differential forms on M can be further decomposed into forms of type (p, q), $0\leqq p, q\leqq n$, of degree $r=p+q$, and the operator d can be decomposed into two components ∂ and $\bar{\partial}$, of types (1, 0) and (0, 1) respectively, with $\partial^2=\bar{\partial}^2=0$. The complex form of the Poincaré Lemma, for $\bar{\partial}$ say, states that a $\bar{\partial}$-closed form of type (p, q) with $q>0$ is locally $\bar{\partial}$-exact. This theorem was proved by Cartan-Grothendieck. The Cartan argument (see [**1**]) uses potential theory to reduce the proof for general forms (with differentiable coefficients) to the case of forms with real analytic coefficients, for which the proof is classical. An outline of this argument was also given by Spencer [**3**] who proves the lemma for the real analytic case by means of a local operator κ of type $(0, -1)$ satisfying

$$\bar{\partial}\kappa\phi + \kappa\bar{\partial}\phi = \phi \tag{1.2}$$

for real analytic forms ϕ of type (p, q) with $q>0$.

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An inductive proof along classical lines was given by Serre [4]. It is the purpose of this note to give an explicit construction for an operator κ satisfying (1.2) defined on all forms suitably bounded in an open neighborhood of an arbitrary point, thereby affording a constructive proof of the complex Poincaré Lemma, such that the form $\kappa\phi$ can be estimated in terms of the form ϕ. The construction is based on a generalization of a formula given by Newlander and Nirenberg [2, Equations (2.4)–(2.6)] for a function f which is a solution of a system of differential equations which may be written as $\bar{\partial} f=\phi$, where ϕ is a differential form of type (0, 1) with $\bar{\partial}\phi=0$.

2. **Preliminary calculations.** Let f be a function which is defined and differentiable (i.e. C^∞) in the polycylinder P: $|z^k|<\rho$, $k=1, \cdots, n$, of complex Euclidean n-space, and suppose further that f and its derivatives (of arbitrary order) with respect to the variables $\bar{z}^k$, $k=1, \cdots, n$, are bounded in P or, briefly that f is admissible[2] in P. Let $T^{\bar{j}}f$, $j=1, \cdots, n$, denote the function given by

$$T^{\bar{j}}f(z^1, \bar{z}^1, \cdots, z^n, \bar{z}^n)$$

$$(2.1)\qquad = \frac{1}{2\pi i}\iint_{|\tau|<\rho} f(z^1, \bar{z}^1, \cdots, z^{j-1}, \bar{z}^{j-1}, \tau, \bar{\tau}, z^{j+1}, \bar{z}^{j+1}, \cdots, z^n, \bar{z}^n) \cdot \frac{d\bar{\tau}d\tau}{z^j - \tau}.$$

Then, as is well known $T^{\bar{j}}f$ is also differentiable and admissible in P, and with $\partial_{\bar{k}}=\partial/\partial\bar{z}^k$, satisfies

$$(2.2)\qquad \partial_{\bar{k}}T^{\bar{j}}f = \begin{cases} T^{\bar{j}}\partial_{\bar{k}}f, & k \neq j, \\ f, & k = j. \end{cases}$$

The operator $T^{\bar{j}}$ is clearly linear.

Next, let $\beta=(\beta_1, \cdots, \beta_q)$ be a set of q distinct integers, $1\leqq\beta_r\leqq n$, $0<q\leqq n$, and let

$$(2.3)\qquad L(q;\beta)f = \sum_{s=0}^{n-q} \frac{(-1)^s}{q(q+1)\cdots(q+s)} \sum_{(j_1,\cdots,j_s;\beta)} T^{\bar{j}_1}\partial_{\bar{j}_1}\cdots T^{\bar{j}_s}\partial_{\bar{j}_s}f,$$

where $(j_1, \cdots, j_s; \beta)$ denotes a set of s distinct integers j_l, with $1\leqq j_l\leqq n$, $j_l\notin\beta$, and the sum is taken over all possible selections. Then $L(q; \beta)f$ is differentiable and admissible in P, and

$$(2.4)\qquad \partial_{\bar{k}}L(q;\beta)f = \begin{cases} L(q+1; k+\beta)\partial_{\bar{k}}f, & k \notin \beta, \\ L(q; \beta)\partial_{\bar{k}}f, & k \in \beta, \end{cases}$$

[2] The construction remains valid when the word "admissible" is interpreted to mean that f has a finite Hölder norm: $f\in\tilde{C}^{n-q+\alpha}$, $0<\alpha<1$ (see [2]).

where $k+\beta$ denotes the set of $q+1$ distinct integers $(k, \beta_1, \cdots, \beta_q)$. The case $k\in\beta$ is trivial, and is the only possible case if $q=n$. The case $k\notin\beta$ is verified by grouping those terms $(j_1, \cdots, j_s; \beta)$ for which some $j_l=k$ (in which case $s>0$ and the remaining j's are different from k) and those terms $(j_1, \cdots, j_s; \beta)$ for which no $j_l=k$ (in which case $s<n-q$) to obtain

$$\begin{aligned}\partial_k L(q;\beta)f &= \sum_{s=1}^{n-q} \frac{(-1)^s s}{q(q+1)\cdots(q+s)} \sum_{(j_1,\cdots,j_{s-1};k+\beta)} T^{\bar{j}_1}\partial_{\bar{j}_1}\cdots T^{\bar{j}_{s-1}}\partial_{\bar{j}_{s-1}}\partial_{\bar{k}} f \\ &\quad + \sum_{s=0}^{n-q-1} \frac{(-1)^s}{q(q+1)\cdots(q+s)} \sum_{(j_1,\cdots,j_s;k+\beta)} T^{\bar{j}_1}\partial_{\bar{j}_1}\cdots T^{\bar{j}_s}\partial_{\bar{j}_s}\partial_{\bar{k}} f \\ &= L(q+1;k+\beta)\partial_{\bar{k}} f\end{aligned}$$

since

$$\frac{(-1)^{s+1}(s+1)}{q(q+1)\cdots(q+s+1)} + \frac{(-1)^s}{q(q+1)\cdots(q+s)} = \frac{(-1)^s}{(q+1)\cdots(q+1+s)}.$$

Another formula which will be needed is

$$(2.5) \qquad qL(q;\beta)f = \begin{cases} f - \sum_{(m;\beta)} L(q+1;m+\beta)T^{\bar{m}}\partial_{\bar{m}} f, & 0<q<n, \\ f, & q=n. \end{cases}$$

3. **Construction of the operator κ.** Let m_0 be an arbitrary point of a complex analytic manifold M of complex dimension n and choose local complex analytic coordinates $z^1, \cdots, z^n$ in a neighborhood of m_0 such that m_0 corresponds to the origin. Let V be the subneighborhood of points whose coordinates satisfy $|z^k|<\rho$, $k=1, \cdots, n$, for some fixed choice of $\rho>0$, sufficiently small. The operator κ will be defined for all admissible forms on V.

A differential form ϕ on V of type (p, q) can be expressed as

$$(3.1) \qquad \phi = \sum'_{(\alpha_1,\cdots,\alpha_p),(\beta_1,\cdots,\beta_q)} \phi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q} dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar{z}^{\beta_1}\wedge\cdots\wedge d\bar{z}^{\beta_q}$$

where $(\alpha_1, \cdots, \alpha_p)$ and $(\beta_1, \cdots, \beta_q)$ are selections of p and q, respectively, distinct integers between 1 and n, and the sum is extended over all possible selections except that only one selection is used from each group of selections differing only by permutations of the selected integers. The sum is independent of the particular choice since the forms $dz^{\alpha_1}\wedge \cdots \wedge dz^{\alpha_p}$ and $d\bar{z}^{\beta_1}\wedge \cdots \wedge d\bar{z}^{\beta_q}$ and the coefficients $\phi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}$, which are differentiable functions on V, are skew-symmetric in their indices. In particular, for $q>0$,

$$\sum_{r=1}^{q} (-1)^{r-1}\phi_{\alpha_1\cdots\alpha_p\bar{\beta}_r\bar{\beta}_1\cdots\bar{\beta}_{r-1}\bar{\beta}_{r+1}\cdots\bar{\beta}_q} = q\phi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}. \tag{3.2}$$

Two forms of the same type are added by adding their coefficients with like indices. The form $\bar{\partial}\phi$ of type $(p, q+1)$ is determined by the coefficients

$$(\bar{\partial}\phi)_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_{q+1}} = \sum_{r=1}^{q+1} (-1)^{p+r-1}\partial_{\bar{\beta}_r}\phi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_{r-1}\bar{\beta}_{r+1}\cdots\bar{\beta}_{q+1}} \tag{3.3}$$

if $0\leqq q<n$, while $\bar{\partial}\phi=0$ if $q=n$. A differential form ϕ on V will be called admissible in V if its coefficients are admissible, as defined in §2, in V. If ϕ is admissible in V, then so is $\bar{\partial}\phi$.

If ϕ is an admissible form of type (p, q) with $q>0$, let

$$(\kappa\phi)_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_{q-1}} = (-1)^p \sum_{(m;\beta)} L(q; m + \beta)T^{\bar{m}}\phi_{\alpha_1\cdots\alpha_p\bar{m}\bar{\beta}_1\cdots\bar{\beta}_{q-1}}, \tag{3.4}$$

where the operators L and T are defined for functions on V as in §2, and where $\beta=(\beta_1, \cdots, \beta_{q-1})$. It is easily seen that the functions (3.4) are skew-symmetric in the appropriate indices and therefore determine a form, which will be denoted by $\kappa\phi$, of type $(p, q-1)$ and that $\kappa\phi$ is admissible in V.

In order to prove the Formula (1.2) for the operator κ, it is sufficient to verify that

$$(\bar{\partial}\kappa\phi)_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q} + (\kappa\bar{\partial}\phi)_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q} = \phi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q} \tag{3.5}$$

for arbitrary indices $(\alpha_1, \cdots, \alpha_p)$, $(\beta_1, \cdots, \beta_q)$. First suppose that $0<q<n$. Then, by definition of κ,

$$\begin{aligned}(\kappa\bar{\partial}\phi)&_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}\\ &= (-1)^p \sum_{(m;\beta)} L(q+1; m+\beta)T^{\bar{m}}(\bar{\partial}\phi)_{\alpha_1\cdots\alpha_p\bar{m}\bar{\beta}_1\cdots\bar{\beta}}\\ &= \sum_{(m;\beta)} L(q+1; m+\beta)T^{\bar{m}}\Big\{\partial_{\bar{m}}\phi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}\\ &\qquad - \sum_{r=1}^{q}(-1)^{r-1}\partial_{\bar{\beta}_r}\phi_{\alpha_1\cdots\alpha_p\bar{m}\bar{\beta}_1\cdots\bar{\beta}_{r-1}\bar{\beta}_{r+1}\cdots\bar{\beta}_q}\Big\}.\end{aligned} \tag{3.6}$$

On the other hand, if $\beta^r=(\beta_1, \cdots, \beta_{r-1}, \beta_{r+1}, \cdots, \beta_q)$, then the range of $(m; \beta^r)$ includes the range of $(m; \beta)$ and, in addition, the value $m=\beta_r$, so

$$
(3.7)\quad
\begin{aligned}
(\bar\partial\kappa\phi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} &= \sum_{r=1}^{q} (-1)^{p+r-1}\partial_{\bar\beta_r}(\kappa\phi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_{r-1}\bar\beta_{r+1}\cdots\bar\beta_q} \\
&= \sum_{r=1}^{q} (-1)^{r-1}\partial_{\bar\beta_r}\left\{\sum_{(m;\beta^r)} L(q;m+\beta^r)T^{\bar m}\phi_{\alpha_1\cdots\alpha_p\bar m\bar\beta_1\cdots\bar\beta_{r-1}\bar\beta_{r+1}\cdots\bar\beta_q}\right\} \\
&= \sum_{r=1}^{q}\sum_{(m;\beta)} (-1)^{r-1}L(q+1;m+\beta)T^{\bar m}\partial_{\bar\beta_r}\phi_{\alpha_1\cdots\alpha_p\bar m\bar\beta_1\cdots\bar\beta_{r-1}\bar\beta_{r+1}\cdots\bar\beta_q} \\
&\quad + \sum_{r=1}^{q} (-1)^{r-1}L(q;\beta)\phi_{\alpha_1\cdots\alpha_p\bar\beta_r\bar\beta_1\cdots\bar\beta_{r-1}\bar\beta_{r+1}\cdots\bar\beta_q}
\end{aligned}
$$

by (2.4) and (2.2). The last sum can be rewritten, by means of (3.2) and (2.5) as

$$\phi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} - \sum_{(m;\beta)} L(q+1;m+\beta)T^{\bar m}\partial_{\bar m}\phi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}$$

so that the sum of (3.6) and (3.7) is indeed $\phi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}$, as was to be shown. The formula is also true for $q=n$, since $\kappa\bar\partial\phi=0$, and the computation analogous to (3.7) yields $\bar\partial\kappa\phi=\phi$.

4. The case $q=0$. In the case of a real differentiable manifold, any operator of degree -1 must vanish on forms of degree 0 (functions). The simplest choice [3] of an operator k satisfying (1.1) for forms of positive degree gives

$$(4.1)\qquad kdf = f - f(m_0)$$

for functions f on V, which is in agreement with the fact that a d-closed function on V is constant, or $f\equiv f(m_0)$. An analogous result holds for the operator κ constructed in §3. A $\bar\partial$-closed function on V is holomorphic, since it cannot depend on the variables $\bar z^1, \cdots, \bar z^n$, and a form of type $(p, 0)$ is called a holomorphic p-form if its coefficients in an expansion of the type (3.1) are holomorphic. From (3.3) it is clear that a form of type $(p, 0)$ is holomorphic if and only if it is $\bar\partial$-closed. For admissible forms ϕ of type $(p, 0)$, the operator κ satisfies

$$(4.2)\qquad \kappa\bar\partial\phi = \phi - J\phi$$

where $J\phi$ is a holomorphic p-form which coincides with ϕ if ϕ is itself holomorphic. In fact, let $J\phi$ be defined by (4.2); that is, let

$$
(4.3)\quad
\begin{aligned}
(J\phi)_{\alpha_1\cdots\alpha_p} &= \phi_{\alpha_1\cdots\alpha_p} - (\kappa\bar\partial\phi)_{\alpha_1\cdots\alpha_p} \\
&= \phi_{\alpha_1\cdots\alpha_p} - \sum_{m} L(1;m)T^{\bar m}\partial_{\bar m}\phi_{\alpha_1\cdots\alpha_p}.
\end{aligned}
$$

Then, by (2.4) and (2.5) for the case $q=1$,

$$\partial_{\bar{k}}(J\phi)_{\alpha_1\cdots\alpha_p} = \partial_{\bar{k}}\phi_{\alpha_1\cdots\alpha_p} - \sum_{(m;k)} L(2;m+k)T^{\bar{m}}\partial_{\bar{m}}\partial_{\bar{k}}\phi_{\alpha_1\cdots\alpha_p} - L(1;k)\partial_{\bar{k}}\phi_{\alpha_1\cdots\alpha_p} = 0$$

for $k=1, \cdots, n$. Thus $(J\phi)_{\alpha_1\cdots\alpha_p}$ is a holomorphic function and $J\phi$ a holomorphic p-form on V.

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