ON THE NONCONVERGENCE OF FOURIER SERIES

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1. Let the Fourier series of an even integrable function $\phi(t)$, periodic with period 2π, be

$$\phi(t) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos nt. \tag{1.1}$$

The undermentioned convergence criteria A and B, for (1.1), have been given by Wang [2].

THEOREM A. *If*

$$\text{(i)} \qquad \int_0^t \phi(u)du = o(t^k), \qquad \text{as } t \to 0,\ k > 1,$$

and

$$\text{(ii)} \qquad a_n > -Kn^{-1/k},$$

for some $K>0$, *then the series* (1.1), *at* $t=0$, *converges.*

THEOREM B. *If*

$$\text{(i)}' \qquad \int_0^t \phi(u)du = o\left(t/\log\frac{1}{t}\right), \qquad \text{as } t \to 0,$$

and

$$\text{(ii)}' \qquad a_n > -K \log n/n,$$

for some $K>0$, *then the series* (1.1), *at* $t=0$, *converges.*

Wang [2] has also framed examples showing that k in condition (i) of Theorem A cannot be replaced by any $k'<k$ and condition (ii)$'$ in Theorem B cannot be replaced by the condition

$$\text{(ii)}'' \qquad a_n = o\{n^{-1}(\log n)^2\}.$$

Hsiang [1] has recently tried to bridge the gap existing between conditions (ii)$'$ and (ii)$''$ by framing examples to prove the following theorem:

THEOREM C. *There exists an even function* $\phi(t)$, *satisfying* (i)$'$, *whose Fourier series diverges at* $t=0$, *while*

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(ii)''' $$a_n = o\{(\log n)^{1+\eta} n^{-1}\}, \qquad \text{for any } \eta > 0.$$

The object of the present paper is to prove by framing examples that Theorems A and B of Wang are best possible in the sense that neither o's in conditions (i) and (i)' can be replaced by O's, nor conditions (ii) and (ii)' can be replaced by

(iii) $a_n = O(n^{-1/k}\rho_n)$ and (iii)' $a_n = O(\log n \cdot n^{-1}\rho_n)$,

respectively, where $\{\rho_n\}$ is an arbitrarily chosen sequence of numbers, tending to infinity with n, however slowly.

Obviously the implication of Example 4, of this paper, showing the best possibility of (ii)', is of more far-reaching character than that of Hsiang.

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2.1. We shall prove the following theorems:

THEOREM 1. *There exists an even, integrable, periodic function* $\phi(t)$, *such that*

$$\int_0^t \phi(u)du = O(t^k), \qquad \text{as } t \to 0,\ k > 1,$$

and $a_n = O(n^{-1/k})$, *whose Fourier series, at* $t=0$, *does not converge.*

THEOREM 2. *There exists an even integrable periodic function* $\phi(t)$, *such that* (i) *is satisfied and* $a_n = O(n^{-1/k}\rho_n)$, $\{\rho_n\}$ *being any arbitrarily chosen sequence of numbers, tending to infinity with* n, *however slowly, whose Fourier series, at* $t=0$, *diverges.*

THEOREM 3. *There exists an even integrable periodic function* $\phi(t)$' *such that*

$$\int_0^t \phi(u)du = O\left(t/\log \frac{1}{t}\right), \qquad \text{as } t \to 0,$$

and $a_n = O(\log n/n)$, *whose Fourier series, at* $t=0$, *does not converge.*

THEOREM 4. *There exists an even, periodic and integrable function* $\phi(t)$, *such that* (i)' *is satisfied and* $a_n = O(\log n \cdot n^{-1}\rho_n)$, $\{\rho_n\}$ *being any arbitrarily chosen sequence of numbers, tending to infinity with* n, *however slowly, whose Fourier series diverges at* $t=0$.

2.2. In order to prove Theorem 1 we frame the following example:

EXAMPLE 1. *Let the sequences* $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *be chosen such that*

$$\lambda_n = (n+1)!,$$

$$\alpha_n = \frac{\pi}{2}\,[[(\lambda_n)^{1-1/k}]](\lambda_n)^{-1},$$

where the symbol $[[p]]$ *denotes the integral part of* p *if* $[p]$ *is odd and the next integer greater than* $[p]$ *if* $[p]$ *is even,* $[p]$ *being the integral part of* p.[1] *The interval* $(3\alpha_n, \alpha_n)$ *will be denoted by* I_n, *for* $n=1, 2, \cdots$.

We now define an even function $\phi(t)$, *such that*

$$\phi(t) = (-1)^n \sin(\lambda_n t),$$

for values of t *lying in* I_n, *and* $\phi(t)=0$, *everywhere else in* $(0, \pi)$.

It is easy to see that $\phi(t)$, being bounded, is integrable (L) over $(0, \pi)$. Also

$$\int_{I_i} \phi(u)du = (-1)^{i+1}(\lambda_i)^{-1}(\cos \lambda_i t)_{t=\alpha_i}^{3\alpha_i} = 0.$$

Hence if t lies in I_p, then

$$\int_0^t \phi(u)du = (-1)^{p+1}\cos(\lambda_p t)/\lambda_p = O(t^k).$$

Again

$$a_n = \frac{2}{\pi}\int_0^\pi \phi(t)\cos nt\,dt$$

$$(2.2.1)\qquad = \frac{2}{\pi}\sum_{i=1}^{\infty}(-1)^i\int_{I_i}\sin\lambda_i t\cdot\cos nt\,dt$$

$$= \sum_{i\leqq\zeta_n} J_i + J_{\zeta_n+1} + \sum_{i>\zeta_n+1} J_i, \text{ say,}$$

where ζ_n is an integer such that $\lambda_{\zeta_n} \leqq n < \lambda_{\zeta_n+1}$. Since the total variation of $\sin(\lambda_i t)$ in I_i is $O(\lambda_i\alpha_i)$, it follows by the second mean value theorem that

$$\sum_{i\leqq\zeta_n} J_i = \sum_{i\leqq\zeta_n} O(\lambda_i\alpha_i/n)$$

$$(2.2.2)\qquad = \sum_{i\leqq\zeta_n} O(\{(i+1)!\}^{1-1/k}/n)$$

$$= O(\{(\zeta_n+1)!\}^{1-1/k}/n)$$

$$= O(n^{-1/k}).$$

Next, if $(\zeta_n+2)!-n<n^{1/k}$, then by the second mean value theorem

[1] This notation will be used throughout this paper.

(2.2.3) $$J_{\zeta_n+1} = O(n^{-1}\{(\zeta_n + 2)!\}^{1-1/k}) = O(n^{-1/k}),$$

and if $(\zeta_n+2)!-n \geqq n^{1/k}$, then

(2.2.4) $$J_{\zeta_n+1} = O(1/\{(\zeta_n + 2)! - n\}) = O(n^{-1/k}).$$

Also

(2.2.5) $$\begin{aligned}\sum_{i>\zeta_n+1} J_i &= \sum_{i\geqq\zeta_n+2} O(1/\{(i+1)! - n\}) \\ &= \sum_{i\geqq\zeta_n+2} O(1/\{i(i)!\}) = O(1/n).\end{aligned}$$

Hence it follows from (2.2.1) $\cdots$ (2.2.5) that $a_n = O(n^{-1/k})$.

We now show that S_n does not converge as $n\to\infty$, where S_n denotes the nth partial sum of the Fourier series of $\phi(t)$, at $t=0$. By well known arguments

$$\begin{aligned}S_{\lambda_n} &= \frac{2}{\pi}\int_0^{\pi} \phi(t)t^{-1}\sin\lambda_n t dt + o(1) \\ &= \frac{2}{\pi}\sum_{i=1}^{\infty}(-1)^i\int_{I_i} t^{-1}\sin\lambda_i t\cdot\sin\lambda_n t dt + o(1) \\ &= \sum_{i=1}^{\infty} G_i + o(1), \text{ say.}\end{aligned}$$

It follows by the second mean value theorem that

(2.2.6) $$\begin{aligned}\sum_{i<n} G_i &= \sum_{i=1}^{n-1} O(1/\{\alpha_i(\lambda_n - \lambda_i)\}) \\ &= O(n/\{\alpha_{n-1}(\lambda_n - \lambda_{n-1})\}) \\ &= o(1),\end{aligned}$$

and, since the total variation of $\sin\lambda_n t$ in I_i is $O(\lambda_n\alpha_i)$, that

(2.2.7) $$\sum_{i>n} G_i = \sum_{i=n+1}^{\infty} O(\lambda_n\alpha_i/(\lambda_i\alpha_i)) = o(1).$$

Finally

(2.2.8) $$\begin{aligned}G_n &= (-1)^n\frac{2}{\pi}\int_{I_n}\sin^2\lambda_n t\cdot t^{-1}dt \\ &= \frac{(-1)^n}{\pi}\int_{I_n}(1/t - \cos(2\lambda_n t)/t)dt \\ &= \frac{(-1)^n}{\pi}\log 3 + o(1).\end{aligned}$$

Collecting the results (2.2.6) $\cdots$ (2.2.8), we get

$$S_{\lambda_n} = \frac{(-1)^n}{\pi} \log 3 + o(1),$$

which oscillates between $(\log 3)/\pi$ and $-(\log 3)/\pi$, as $n \to \infty$.

This establishes Theorem 1.

2.3. Theorem 2 will be proved by means of the following example:

EXAMPLE 2. *Let*

$$\lambda_n = (n+1)!,$$
$$d_p = [[(\rho_n)^{1/2}]],$$

for all values of p *such that* $\zeta_n \leqq p \leqq \zeta_n + 1$, *where* $\{\zeta_n\}$ *is the sequence of integers such that* $\lambda_{\zeta_n} \leqq n < \lambda_{\zeta_n+1}$, *and* $\{\rho_n\}$ *is an arbitrarily chosen sequence of numbers tending to infinity with* n, *however slowly, and let*

$$\alpha_{2n} = \frac{\pi}{2} [[(\lambda_{2n})^{1-1/k}]](\lambda_{2n})^{-1},$$

$$\alpha_{2n-1} = \frac{\pi}{2} [[(\lambda_{2n})^{1-1/k} d_{2n}]](\lambda_{2n})^{-1}.$$

The interval $(\alpha_{2n}, \alpha_{2n-1})$ *will be denoted by* I_{2n}.

We now define an even function $\phi(t)$, *such that*

$$\phi(t) = \sin (\lambda_{2n} d_{2n} t),$$

for t *lying in* I_{2n}, *and* $\phi(t) = 0$ *everywhere else in* $(0, \pi)$.

It is easy to see that $\phi(t)$, being bounded, is integrable (L), and for t lying in I_{2n},

$$\int_0^t \phi(u)du = O\{1/(\lambda_{2n} d_{2n})\} = o(t^k).$$

Next

$$a_n = \sum_{2i \leq \zeta_n} J_{2i} + J_{\zeta_n+1} + \sum_{2i > \zeta_n+1} J_{2i},$$

where

$$J_{2i} = \frac{2}{\pi} \int_{I_{2i}} \sin (\lambda_{2i} d_{2i} t) \cos nt dt.$$

The term J_{ζ_n+1} exists only if ζ_n is odd. Proceeding exactly as in Example 1 we get

$$\sum_{2i \leq \zeta_n} J_{2i} = O\{((\zeta_n + 1)!)^{1-1/k} n^{-1} (d_{\zeta_n})^2\} = O(n^{-1/k} \rho_n).$$

Supposing that ζ_n is odd we have, if $(\zeta_n+2)!d_{\zeta_n+1}-n<n^{1/k}/\rho_n$, then

$$J_{\zeta_n+1} = O(n^{-1}\{(\zeta_n + 2)!\}^{1-1/k}(d_{\zeta_n} + 1)^2) = O(n^{-1/k}\rho_n)$$

and if $(\zeta_n+2)!d_{\zeta_n+1}-n \geqq n^{1/k}/\rho_n$, then

$$J_{\zeta_n+1} = O(1/\{\lambda_{\zeta_n+1}d_{\zeta_n+1} - n\}) = O(n^{-1/k}\rho_n).$$

Also it follows by easy calculations that

$$\sum_{2i>\zeta_n+1} J_{2i} = O(1/n),$$

which yields that $a_n = O(n^{-1/k}\rho_n)$.

Writing

$$S_{\lambda_{2n}d_{2n}} = \sum_{i=1}^{\infty} G_{2i} + o(1),$$

where

$$G_{2i} = \frac{2}{\pi}\int_{I_{2i}} t^{-1} \sin (\lambda_{2i}d_{2i}t)\cdot \sin (\lambda_{2n}d_{2n}t)dt.$$

we obtain easily

$$\sum_{i<n} G_{2i}, \qquad \sum_{i>n} G_{2i} = o(1),$$

as in Example 1, and

$$\begin{aligned} G_{2n} &= \frac{1}{\pi}\int_{I_{2n}} \{t^{-1} - t^{-1} \cos (2\lambda_{2n}d_{2n}t)\}dt \\ &= \frac{1}{\pi} \log (\alpha_{2n-1}/\alpha_{2n}) + o(1) \\ &\simeq A \log \rho_n + o(1), \qquad (A \text{ being a constant}) \\ &\to \infty, \text{ as } \rho_n \to \infty, \end{aligned}$$

which establishes Theorem 2.

2.4. Without entering into the details of illustration, we state Examples 3 and 4 which, following an analysis similar to that used in the illustrations of Examples 1 and 2, will go to prove Theorems 3 and 4, respectively.

Example 3. *Let*

$$\lambda_i = \{(i + 1)!\}!,$$

$$\alpha_i = \frac{\pi}{2} [[\log \lambda_i]](\lambda_i)^{-1},$$

and I_i *denote the interval* $(\alpha_i, 3\alpha_i)$. *We now define an even function* $\phi(t)$, *such that*

$$\phi(t) = (-1)^i \sin \lambda_i t,$$

for values of t *lying in* I_i *and* $\phi(t) = 0$ *everywhere else in* $(0, \pi)$.

EXAMPLE 4. *Let*

$$\lambda_n = \{(n+1)!\}!,$$

$$\alpha_{2n} = \frac{\pi}{2} [[\log \lambda_{2n}]](\lambda_{2n})^{-1},$$

$$\alpha_{2n-1} = \frac{\pi}{2} [[\log \lambda_{2n}]] d_{2n}(\lambda_{2n})^{-1},$$

where

$$d_p = [[(\rho_n)^{1/2}]]$$

for all values of p *such that* $\zeta_n \leqq p \leqq \zeta_n + 1$, *where* $\{\zeta_n\}$ *is the sequence of integers such that* $\lambda_{\zeta_n} \leqq n < \lambda_{\zeta_n+1}$. *Let* I_{2n} *denote the interval* $(\alpha_{2n}, \alpha_{2n-1})$. *We now define an even function* $\phi(t)$, *such that*

$$\phi(t) = \sin(\lambda_{2i} d_{2i} t),$$

for values of t *lying in* I_{2i} *and* $\phi(t) = 0$ *everywhere else in* $(0, \pi)$.

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