ON A FORMULA OF A. C. DIXON

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1. A well known formula of A. C. Dixon [**1**], which is equivalent to the functional identity

$$(1)\qquad \left(1+\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}+\cdots\right)\left(1-\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}-\cdots\right) = 1-\frac{3!}{(1!2!)^3}x^2+\frac{6!}{(2!4!)^3}x^4-\cdots$$

due to Ramanujan [**2**], reads:

$$(2)\qquad \sum_{r=0}^{2n}(-1)^r\binom{2n}{r}^3 = (-1)^n\frac{(3n)!}{(n!)^3}, \quad n = 0, 1, 2, \cdots.$$

In their proper setting, the formulae (1) and (2) belong to the theory of hypergeometric series. We however present below a very simple and elementary approach to (2) leading to some generalizations from which this formula appears in a new light. As is well known, the first elementary proof of (2) is due to H. W. Richmond [**5**]. For two other elementary proofs of more recent origin, we refer to the papers[1] [**3**] and [**6**].

2. We can clearly set up the identity

$$(3)\qquad \sum_{r=0}^{n}\binom{n}{r}^p\alpha^{n-r}\beta^r = \sum_{r=0}^{[n/2]}C_{n,r}^{(p)}\binom{n-r}{r}(\alpha+\beta)^{n-2r}(\alpha\beta)^r,$$

where the coefficients $C_{n,r}^{(p)}$ are successively determined by the equations

$$(4)\qquad \binom{n}{k}^p = \sum_{r=0}^{k}C_{n,r}^{(p)}\binom{n-r}{r}\binom{n-2r}{k-r}, \qquad k = 0, 1, \cdots, [n/2],$$

expressing the equality of the coefficients of $\alpha^{n-k}\beta^k$ on both sides of (3). In fact, (3) and (4) are completely equivalent.

Observe that (4) may be rewritten in the form

$$(5)\qquad \binom{n}{k}^p = \sum_{r=0}^{k}C_{n,r}^{(p)}\binom{k}{r}\binom{n-r}{k}.$$

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[1] These were not available to the author, except for their report in the Mathematical Reviews.

Using Vandermonde's identity, we can express the right member in (5) as a repeated sum obtaining

$$\binom{n}{k}^p = \sum_{r=0}^{k} C_{n,r}^{(p)} \binom{k}{r} \sum_{s=r}^{k} \binom{n-k}{s} \binom{k-r}{k-s}.$$

Changing the order of summation on the right and observing that

$$\binom{k}{r}\binom{k-r}{k-s} = \binom{k}{s}\binom{s}{r},$$

we have

$$\binom{n}{k}^p = \sum_{s=0}^{k} \binom{k}{s}\binom{n-k}{s} \sum_{r=0}^{s} C_{n,r}^{(p)} \binom{s}{r}.$$

On multiplying by

$$\binom{n}{k}$$

and observing that

$$\binom{n}{k}\binom{n-k}{s} = \binom{n}{s}\binom{n-s}{k},$$

we have

$$\binom{n}{k}^{p+1} = \sum_{s=0}^{k} \binom{k}{s}\binom{n-s}{k} \cdot \binom{n}{s} \sum_{r=0}^{s} \binom{s}{r} C_{n,r}^{(p)}. \tag{6}$$

Comparison of (5) and (6) establishes for the $C_{n,r}^{(p)}$ the following recursion with respect to p:

$$C_{n,s}^{(p+1)} = \binom{n}{s} \sum_{r=0}^{s} \binom{s}{r} C_{n,r}^{(p)}, \qquad 0 \leqq s \leqq [n/2]. \tag{7}$$

Now for $p=1$, (3) gives (binomial identity !):

$$C_{n,r}^{(1)} = \binom{0}{r}, \tag{8}$$

whence (7) yields, by iteration, the following explicit representation for the $C_{n,r}^{(p)}$ ($p=1, 2, 3, \cdots$):

$$C_{n,r}^{(p)} = \sum_{0 \leq r_1 \leq \cdots \leq r_p \leq r} \binom{0}{r_1}\binom{r_2}{r_1} \cdots \binom{r_p}{r_{p-1}}\binom{r_p}{r}\binom{n}{r_1} \cdots \binom{n}{r_p}. \tag{9}$$

In particular, we have

$$C_{n,r}^{(2)} = \binom{n}{r}, \qquad C_{n,r}^{(3)} = \binom{n+r}{r}\binom{n}{r}. \tag{10}$$

In deriving the latter formula in (10), Vandermonde's identity has been used. Taking $p=3$, $\alpha=1$, $\beta=-1$ and replacing n by $2n$ in (3), we obtain (2). The case $p = 3$ of (3) has been derived by P. A. MacMahon [**4**] by another procedure which is not altogether elementary.

3. If, in (7), which evidently holds for arbitrary p, we write $-(p+1)$ for p and define $d_{n,r}^{(p)}$ with

$$C_{n,r}^{(-p)} = (-1)^r d_{n,r}^{(p)}, \tag{11}$$

it readily takes the form

$$\binom{n}{s}^{-1} d_{n,s}^{(p)} = \Delta^s d_{n,0}^{(p+1)}.$$

Here we have used the notation of finite differences with Δ operating on the index s in $d_{n,s}^{(p)}$. Now

$$d_{n,s}^{(p+1)} = E^s d_{n,0}^{(p+1)} = (1+\Delta)^s d_{n,0}^{(p+1)} = \sum_{r=0}^{s} \binom{s}{r} \Delta^r d_{n,0}^{(p+1)}.$$

Herewith we secure the following recursion for the $d_{n,r}^{(p)}$ with respect to p:

$$d_{n,s}^{(p+1)} = \sum_{r=0}^{s} \binom{s}{r}\binom{n}{r}^{-1} d_{n,r}^{(p)}, \qquad 0 \leqq s \leqq [n/2]. \tag{12}$$

We have, according to (8) and the definition (11),

$$d_{n,r}^{(-1)} = \binom{0}{r}, \tag{13}$$

whence (12) yields, by iteration, the following explicit representation for the $d_{n,r}^{(p)}$ ($p=0, 1, \cdots$):

$$d_{n,r}^{(p)} = \sum_{0\leqq r_0 \leqq \cdots \leqq r_p \leqq r} \binom{0}{r_0}\binom{r_1}{r_0} \cdots \binom{r_p}{r_{p-1}}\binom{r}{r_p}\binom{n}{r_0}^{-1} \cdots \binom{n}{r_p}^{-1}. \tag{14}$$

In particular, we have

$$d_{n,r}^{(0)} = 1, \qquad d_{n,r}^{(1)} = \binom{n+1}{r}\binom{n}{r}^{-1}. \tag{15}$$

In deriving the latter formula in (15), we have made use of the relations

$$\binom{n}{r}\binom{r}{s} = \binom{n}{s}\binom{n-s}{r-s}, \quad \binom{n+1}{r} = \sum_{s=0}^{r}\binom{n-s}{r-s}, \qquad s \leqq r.$$

We thus have the interesting identity:

$$\begin{aligned}\sum_{r=0}^{n}\binom{n}{r}^{-1}&\alpha^{n-r}\beta^{r} \\ &= \sum_{r=0}^{[n/2]}(-1)^r\binom{n+1}{r}\binom{n}{r}^{-1}\binom{n-r}{r}(\alpha+\beta)^{n-2r}(\alpha\beta)^r.\end{aligned} \tag{16}$$

This gives, as a special case with $\alpha=1$, $\beta=-1$ and $2n$ in place of n, the neat formula

$$\sum_{r=0}^{2n}(-1)^r\Big/\binom{2n}{r} = (2n+1)/(n+1), \qquad n = 0, 1, 2, \cdots. \tag{17}$$

This is analogous to (2), but not so interesting. In fact it is a particular case of the more readily derivable formula

$$\sum_{r=0}^{n}(-1)^r\Big/\binom{m-1}{r} = \frac{m}{m+1}\left\{1+(-1)^n\Big/\binom{m}{n+1}\right\}, \qquad m > n \geqq 0. \tag{18}$$

For the proof of (18), we need only notice the relations

$$m\binom{m-1}{r} = (m-r)\binom{m}{r} = (r+1)\binom{m}{r+1}, \qquad 0 \leqq r \leqq n.$$

References

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