

ON A FORMULA OF A. C. DIXON

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1. A well known formula of A. C. Dixon [1], which is equivalent to the functional identity

$$(1) \quad \left(1 + \frac{x}{(1!)^3} + \frac{x^2}{(2!)^3} + \dots\right) \left(1 - \frac{x}{(1!)^3} + \frac{x^2}{(2!)^3} - \dots\right) \\ = 1 - \frac{3!}{(1!2!)^3} x^2 + \frac{6!}{(2!4!)^3} x^4 - \dots$$

due to Ramanujan [2], reads:

$$(2) \quad \sum_{r=0}^{2n} (-1)^r \binom{2n}{r}^3 = (-1)^n \frac{(3n)!}{(n!)^3}, \quad n = 0, 1, 2, \dots$$

In their proper setting, the formulae (1) and (2) belong to the theory of hypergeometric series. We however present below a very simple and elementary approach to (2) leading to some generalizations from which this formula appears in a new light. As is well known, the first elementary proof of (2) is due to H. W. Richmond [5]. For two other elementary proofs of more recent origin, we refer to the papers¹ [3] and [6].

2. We can clearly set up the identity

$$(3) \quad \sum_{r=0}^n \binom{n}{r}^p \alpha^{n-r} \beta^r = \sum_{r=0}^{[n/2]} C_{n,r}^{(p)} \binom{n-r}{r} (\alpha + \beta)^{n-2r} (\alpha\beta)^r,$$

where the coefficients $C_{n,r}^{(p)}$ are successively determined by the equations

$$(4) \quad \binom{n}{k}^p = \sum_{r=0}^k C_{n,r}^{(p)} \binom{n-r}{r} \binom{n-2r}{k-r}, \quad k = 0, 1, \dots, [n/2],$$

expressing the equality of the coefficients of $\alpha^{n-k}\beta^k$ on both sides of (3). In fact, (3) and (4) are completely equivalent.

Observe that (4) may be rewritten in the form

$$(5) \quad \binom{n}{k}^p = \sum_{r=0}^k C_{n,r}^{(p)} \binom{k}{r} \binom{n-r}{k}.$$

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¹ These were not available to the author, except for their report in the Mathematical Reviews.

Using Vandermonde's identity, we can express the right member in (5) as a repeated sum obtaining

$$\binom{n}{k}^p = \sum_{r=0}^k C_{n,r}^{(p)} \binom{k}{r} \sum_{s=r}^k \binom{n-k}{s} \binom{k-r}{k-s}.$$

Changing the order of summation on the right and observing that

$$\binom{k}{r} \binom{k-r}{k-s} = \binom{k}{s} \binom{s}{r},$$

we have

$$\binom{n}{k}^p = \sum_{s=0}^k \binom{k}{s} \binom{n-k}{s} \sum_{r=0}^s C_{n,r}^{(p)} \binom{s}{r}.$$

On multiplying by

$$\binom{n}{k}$$

and observing that

$$\binom{n}{k} \binom{n-k}{s} = \binom{n}{s} \binom{n-s}{k},$$

we have

$$(6) \quad \binom{n}{k}^{p+1} = \sum_{s=0}^k \binom{k}{s} \binom{n-s}{k} \cdot \binom{n}{s} \sum_{r=0}^s \binom{s}{r} C_{n,r}^{(p)}.$$

Comparison of (5) and (6) establishes for the $C_{n,r}^{(p)}$ the following recursion with respect to p :

$$(7) \quad C_{n,s}^{(p+1)} = \binom{n}{s} \sum_{r=0}^s \binom{s}{r} C_{n,r}^{(p)}, \quad 0 \leq s \leq [n/2].$$

Now for $p=1$, (3) gives (binomial identity !):

$$(8) \quad C_{n,r}^{(1)} = \binom{0}{r},$$

whence (7) yields, by iteration, the following explicit representation for the $C_{n,r}^{(p)}$ ($p=1, 2, 3, \dots$):

$$(9) \quad C_{n,r}^{(p)} = \sum_{0 \leq r_1 \leq \dots \leq r_p \leq r} \binom{0}{r_1} \binom{r_2}{r_1} \dots \binom{r_p}{r_{p-1}} \binom{r_p}{r} \binom{n}{r_1} \dots \binom{n}{r_p}.$$

In particular, we have

$$(10) \quad C_{n,r}^{(2)} = \binom{n}{r}, \quad C_{n,r}^{(3)} = \binom{n+r}{r} \binom{n}{r}.$$

In deriving the latter formula in (10), Vandermonde’s identity has been used. Taking $p=3, \alpha=1, \beta=-1$ and replacing n by $2n$ in (3), we obtain (2). The case $p=3$ of (3) has been derived by P. A. MacMahon [4] by another procedure which is not altogether elementary.

3. If, in (7), which evidently holds for arbitrary p , we write $-(p+1)$ for p and define $d_{n,r}^{(p)}$ with

$$(11) \quad C_{n,r}^{(-p)} = (-1)^r d_{n,r}^{(p)}$$

it readily takes the form

$$\binom{n}{s}^{-1} d_{n,s}^{(p)} = \Delta^s d_{n,0}^{(p+1)}.$$

Here we have used the notation of finite differences with Δ operating on the index s in $d_{n,s}^{(p)}$. Now

$$d_{n,s}^{(p+1)} = E^s d_{n,0}^{(p+1)} = (1 + \Delta)^s d_{n,0}^{(p+1)} = \sum_{r=0}^s \binom{s}{r} \Delta^r d_{n,0}^{(p+1)}.$$

Herewith we secure the following recursion for the $d_{n,r}^{(p)}$ with respect to p :

$$(12) \quad d_{n,s}^{(p+1)} = \sum_{r=0}^s \binom{s}{r} \binom{n}{r}^{-1} d_{n,r}^{(p)}, \quad 0 \leq s \leq [n/2].$$

We have, according to (8) and the definition (11),

$$(13) \quad d_{n,r}^{(-1)} = \binom{0}{r},$$

whence (12) yields, by iteration, the following explicit representation for the $d_{n,r}^{(p)}$ ($p=0, 1, \dots$):

$$(14) \quad d_{n,r}^{(p)} = \sum_{0 \leq r_0 \leq \dots \leq r_p \leq r} \binom{0}{r_0} \binom{r_1}{r_0} \dots \binom{r_p}{r_{p-1}} \binom{r}{r_p} \binom{n}{r_0}^{-1} \dots \binom{n}{r_p}^{-1}.$$

In particular, we have

$$(15) \quad d_{n,r}^{(0)} = 1, \quad d_{n,r}^{(1)} = \binom{n+1}{r} \binom{n}{r}^{-1}.$$

In deriving the latter formula in (15), we have made use of the relations

$$\binom{n}{r} \binom{r}{s} = \binom{n}{s} \binom{n-s}{r-s}, \quad \binom{n+1}{r} = \sum_{s=0}^r \binom{n-s}{r-s}, \quad s \leq r.$$

We thus have the interesting identity:

$$(16) \quad \sum_{r=0}^n \binom{n}{r}^{-1} \alpha^{n-r} \beta^r = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n+1}{r} \binom{n}{r}^{-1} \binom{n-r}{r} (\alpha + \beta)^{n-2r} (\alpha\beta)^r.$$

This gives, as a special case with $\alpha = 1, \beta = -1$ and $2n$ in place of n , the neat formula

$$(17) \quad \sum_{r=0}^{2n} (-1)^r / \binom{2n}{r} = (2n+1)/(n+1), \quad n = 0, 1, 2, \dots$$

This is analogous to (2), but not so interesting. In fact it is a particular case of the more readily derivable formula

$$(18) \quad \sum_{r=0}^n (-1)^r / \binom{m-1}{r} = \frac{m}{m+1} \left\{ 1 + (-1)^n / \binom{m}{n+1} \right\}, \quad m > n \geq 0.$$

For the proof of (18), we need only notice the relations

$$m \binom{m-1}{r} = (m-r) \binom{m}{r} = (r+1) \binom{m}{r+1}, \quad 0 \leq r \leq n.$$

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