2. G. Birkhoff and S. MacLane, *A survey of modern algebra*, MacMillan, 1953.
3. P. R. Halmos, *Algebraic Logic*, I—*Monadic Boolean algebras*, Compositia Math. vol. 12 (1955) pp. 217–249.
4. J. C. C. McKinsey and Alfred Tarski, *The algebra of topology*, Ann. of Math. vol. 45 (1944) pp. 141–191.

UNIVERSITY OF CHICAGO

ON HARMONIC MAPPINGS[1]

JOHANNES C. C. NITSCHE

1. Suppose that the functions $x=x(\alpha, \beta)$, $y=y(\alpha, \beta)$ define a one-to-one harmonic mapping of the unit disc Γ in the α, β-plane ($\alpha+i\beta=\gamma$) onto a convex domain C in the x, y-plane ($x+iy=z$). The origin is assumed to be fixed. Introducing two functions $F(\gamma)$ and $G(\gamma)$ which, in Γ, depend analytically upon the variable γ we may write $z=\operatorname{Re} F(\gamma)+i \operatorname{Re} G(\gamma)$. The purpose of the present paper is (i) to give a new proof of a lemma which, in a special form, was first used by T. Radó [**13**] and which was proved in general by L. Bers (see [**2**, Lemma 3.3]),[2] (ii) to derive an improved value for an important constant first introduced by E. Heinz [**3**]. The proofs will be very simple due to the fact that there is a close connection between univalent harmonic mappings and the minimal surface equation (see e.g. [**11**, footnote 2]) and also the differential equation

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1. \tag{1}$$

The connection with the latter equation was exploited by K. Joergens [**8**] for the study of the solutions of (1). One can, however, proceed one step further by introducing a mapping which was invented by H. Lewy [**10**] for Monge-Ampère equations.

2. Let $z=\operatorname{Re} F(\gamma)+i \operatorname{Re} G(\gamma)$ be a harmonic mapping with the properties mentioned above. Then the expression

---

Presented to the Society August 20, 1956 and November 17, 1956; received by the editors September 10, 1957.

[1] This paper was prepared under Contract N onr-710(16) between the University of Minnesota and the Office of Naval Research.

[2] It has been shown by H. Hopf (cf. [**7**, p. 133 and **5**, pp. 91–92]) that the combination of Heinz's inequality with Schwarz's lemma yields a sharper result.

$$\phi = \frac{1}{2} \operatorname{Im}\left(F\bar{G} + \int_0^\gamma (FG' - F'G)d\gamma \right) \tag{2}$$

may be regarded as a function $\phi(x, y)$ of x and y, defined in C (see K. Joergens [**8**, p. 339]). By a straightforward computation it can be verified that $\phi(x, y)$ is a solution of the Equation (1). In fact, one obtains

$$\begin{gathered} p = -\operatorname{Im} G(\gamma), \qquad q = \operatorname{Im} F(\gamma), \\ r = |G'|^2 \cdot [\operatorname{Im} F'\bar{G}']^{-1}, \qquad s = -[\operatorname{Re} F'\bar{G}'] \cdot [\operatorname{Im} F'\bar{G}']^{-1}, \\ t = |F'|^2 \cdot [\operatorname{Im} F'\bar{G}']^{-1}. \end{gathered} \tag{3}$$

Here p, q, r, s, t are abbreviations for ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{xy}, ϕ_{yy}, as usual. According to a lemma of H. Lewy [**9**], $\operatorname{Im}(F'\bar{G}') = x_\alpha y_\beta - x_\beta y_\alpha \neq 0$ in Γ. It may be assumed that $\operatorname{Im}(F'\bar{G}') > 0$. Then $\phi_{xx} > 0$. Now consider, in C, the functions

$$u = u(x, y) = x + p(x, y), \qquad v = v(x, y) = y + q(x, y) \tag{4}$$

and put $u + iv = w$. For any two points z_1 and z_2 in C the following inequality holds true

$$\begin{aligned} &(x_2 - x_1)[p(x_2, y_2) - p(x_1, y_1)] + (y_2 - y_1)[q(x_2, y_2) - q(x_1, y_1)] \\ &\qquad = \tilde{r}(x_2 - x_1)^2 + 2\tilde{s}(x_2 - x_1)(y_2 - y_1) + \tilde{t}(y_2 - y_1)^2 \geqq 0. \end{aligned} \tag{5}$$

Here $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ stand for the values of r, s, t in a point of the segment connecting z_1 with z_2. Substitute (4) into (5):

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 \leqq (x_2 - x_1)(u_2 - u_1) + (y_2 - y_1)(v_2 - v_1) \tag{6}$$

and hence

$$|z_2 - z_1| \leqq |w_2 - w_1|, \tag{7}$$

equality holding only if $z_1 = z_2$ (see H. Lewy [**10**]). Therefore the mapping (4) is one-to-one and it enlarges distances. Denote by Ω the image domain of C under this mapping. On the other hand, going back to the definitions of x, y and p, q one finds

$$w = F(\gamma) + iG(\gamma) \equiv W(\gamma). \tag{8}$$

That is to say the domain Ω is also the schlicht conformal image of Γ under the mapping function $W(\gamma)$.

3. It is easy to see that, starting out with a solution $\phi(x, y)$ of (1), the inverse mapping $w \to z$ under all circumstances is harmonic. Furthermore it turns out that the expression $f = 2\bar{z} - \bar{w}$ which can be regarded as a function of u and v is an analytic function of w. The

inequality $|df/dw| < 1$ which is satisfied by its derivative has interesting consequences, see [**12**].

4. The lemma in question states that there cannot exist a schlicht harmonic mapping of the unit disc Γ onto the whole z-plane. The proof is obvious since, if C would be the whole z-plane then Ω would have to be the whole w-plane. But, at the same time, Ω is the conformal image of Γ. This is not possible.

5. Suppose now that C, like Γ, is the unit disc $|z| < 1$. E. Heinz [**3**] has established an inequality

$$x_\alpha^2 + x_\beta^2 + y_\alpha^2 + y_\beta^2\big]_{\gamma=0} \geqq \mu. \tag{9}$$

Here his constant μ is independent of the individual harmonic mapping under consideration. Heinz found $\mu \geqq 2 - (8/\pi)\sum_{n=2}^{\infty} n^{-2} = 0.358$. Using the relations derived above one obtains the formula

$$x_\alpha^2 + x_\beta^2 + y_\alpha^2 + y_\beta^2 = \frac{r+t}{2+r+t} \cdot \left|\frac{dW}{d\gamma}\right|^2. \tag{10}$$

Remembering the properties of the mapping (4) we know that Ω contains at least a circle of radius 1. Hence, by Schwarz's lemma, $|dW(0)/d\gamma| \geqq 1$. In fact, the sign of equality cannot hold since $\partial(u, v)/\partial(x, y) = 2 + r + t \geqq 4$. Furthermore $1/2 \leqq (r+t)/(2+r+t) < 1$. Combining these two inequalities we conclude

$$\mu \geqq 1/2. \tag{11}$$

6. We wish to mention that H. Hopf[3] has given another simple proof of the value 1/2 for the constant μ. A similar inequality to (9) holds also for more general univalent mappings, see P. Berg [**1**], E. Heinz [**4**; **5**]. However, remaining with the harmonic mappings: the best value of μ is not known.[4] If one takes the polynomial solution $\phi(x, y) = cx^2/2 + y^2/2c$ then Ω is an ellipse with the semiaxes $1+c$ and $1+1/c$. A computation yields

$$\lim_{c\to\infty} \frac{r+t}{2+r+t} = 1, \qquad \lim_{c\to\infty}\left|\frac{dW(0)}{d\gamma}\right| = \frac{4}{\pi},$$

and hence

[3] In a letter of October 26, 1956.

[4] *Added in proof*: A refinement of the preceding method yields even $\mu \geqq 0.64$, as will be shown elsewhere. Therefore, referring to Richert's example for an upper bound, one knows: $0.64 \leqq \mu \leqq 27/2\pi^2$.

$$\lim_{c\to\infty} [x_\alpha^2 + x_\beta^2 + y_\alpha^2 + y_\beta^2]_{\gamma=0} = 16/\pi^2.$$

By an example of H. E. Richert (cf. E. Hopf [6, p. 802]) it is, however, known that the value $16/\pi^2$ is too large

LITERATURE

1. P. Berg, *On univalent mappings by solutions of linear elliptic partial differential equations*, Trans. Amer. Math. Soc. vol. 84 (1957) pp. 310–318.

2. L. Bers, *Isolated singularities of minimal surfaces*, Ann. of Math. vol. 53 (1951) pp. 364–386.

3. E. Heinz, *Ueber die Loesungen der Minimalflaechengleichung*, Nachr. Akad. Wiss. Goettingen, Math.-Phys. Kl. (1952) pp. 51–56.

4. E. Heinz, *Ueber gewisse elliptische Systeme von Differentialgleichungen zweiter Ordnung mit Anwendung auf die Monge-Ampère'sche Gleichung*, Math. Ann. vol. 131 (1956) pp. 411–428.

5. E. Heinz, *On certain non-linear elliptic differential equations and univalent mappings*, Technical Report No. 63, Stanford University, 1957.

6. E. Hopf, *On an inequality for minimal surfaces* $z=z(x, y)$, Journal of Rational Mechanics and Analysis vol. 3 (1953) pp. 519–522 and 801–802.

7. K. Joergens, *Ueber die Loesungen der Differentialgleichung* $rt-s^2=1$, Math. Ann. vol. 127 (1954) pp. 130–134.

8. ———, *Harmonische Abbildungen und die Differentialgleichung* $rt-s^2=1$, Math. Ann. vol. 129 (1955) pp. 330–344.

9. H. Lewy, *On the nonvanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 689–692.

10. ———, *A priori limitations for solutions of Monge-Ampère equations* II, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 365–374.

11. J. Nitsche, *Ueber eine mit der Minimalflaechengleichung zusammenhaengende analytische Funktion und den Bernsteinschen Satz*, Archiv der Mathematik vol. 7 (1956) pp. 417–419.

12. ———, *Elementary proof of Bernstein's theorem on minimal surfaces*, Ann. of Math. vol. 66 (1957) pp. 543–544.

13. T. Radó, *Zu einem Satze von Bernstein ueber Minimalflaechen im Grossen*, Math. Ztschr. vol. 26 (1927) pp. 559–565.

UNIVERSITY OF MINNESOTA