SOME REMARKS ON THE MULTIPLICATIVE GROUP OF A SFIELD

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Introduction. In this note when K is a sfield then K' will denote the multiplicative group of K. We shall show that if H is any subfield of K or a noncommutative subsfield of the sfield K (with some few exceptions) and if H' is subinvariant in K' then H' is invariant in K' and hence H is either K itself or in the center of K. This result extends the Cartan-Brauer-Hua theorem (cf. [**1**]).

Notation. If M is a subset of K then $Z(M)$ will denote the centralizer of M; this is a sfield, its multiplicative group will be denoted by $Z'(M)$. The normalizer of M in K' will be denoted by $N(M)$; and the normalizer of $N(M)$ by $N^2(M)$. If J is an invariant subgroup of L we shall write $J \triangle L$. If J is subinvariant in L, that is, if J is a member of a composition series of L we shall write $J \triangle \triangle L$.

Conclusions. The first lemma is essentially the argument of [**1**].

Lemma 1. *If M is a subsfield of the sfield K and if x is in $N(M)$ but not in M nor in $Z(M)$ then for all m in $Z(M) \cap M'$, $m+x$ is not in $N(M)$.*

Proof. Since x is not in $Z(M)$ there exists an n in M such that $nx = xn'$ where n' is in M and $n' \neq n$. Then $(m+x)^{-1}n(m+x) = (m+x)^{-1}[nm+xn'+mn'-mn'] = (m+x)^{-1}(nm-mn')+n'$. If the left member were in M then so would be the right member and consequently $(m+x)^{-1}(nm-mn')$; then $(m+x)^{-1}$ and hence x are in M contrary to hypothesis.

Corollary 1 (Cartan-Brauer-Hua Theorem). *The only invariant subsfields of K are K itself and subfields of the center of K.*

Proof. Suppose M is an invariant subsfield of K not equal to K nor contained in the center of K. Then there is a nonzero x not in M' and a nonzero y not in $Z(M)$; and one of the three elements x, y, and xy is in neither M' nor in $Z'(M)$. For if two of these elements were in one of these sfields it would follow that the third is also there. It follows from the lemma that there is a nonzero element outside $N(M)$ contrary to hypothesis.

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LEMMA 2. *If H and M are subfields of K, H not contained in M, and if H' is contained in $N(M)$, then H is contained in $Z(M)$.*

PROOF. If H were not in $Z(M)$, then there would be an x in H, x not in $Z(M)$. Since H is not in M there is a y in H', y not in M. Now the three elements x, y, and xy are all in H' and hence in $N(M)$. One of them is in neither M nor in $Z(M)$ for if any two are in one of these sfields then the third is there also. This is a contradiction of Lemma 1 since the elements $x+1$, $y+1$, and $xy+1$ are also in H' and hence in $N(M)$. We conclude that H is contained in $Z(M)$ as the lemma asserts.

COROLLARY 2. *If M is a sfield which is the centralizer of its center H then $N(M)=N^2(M)=N(H)=N^2(H)$.*

PROOF. H is invariant in $N(M)$ and hence $N(M)$ is contained in $N(H)$. On the other hand M is contained in $N(H)$ and is normal in $N(H)$ since M is the centralizer of H. Hence $N(H)$ is contained in $N(M)$ and therefore $N(H)=N(M)$.

Now suppose there is a y in $N^2(M)$, y not in $N(M)$. Then y transforms H into a conjugate field G not contained in H, but contained in $N(M)$ and invariant in $N(M)$. By Lemma 2, G is contained in $Z(H)=M$ and, since M is in $N(M)$, G is invariant in M. But then by the Cartan-Brauer-Hua theorem since G is not contained in H, G must be equal to M. It follows that M is Abelian and hence equal to H equal to G contrary to the fact that y was chosen out of $N(M)$. We conclude that $N(M)=N^2(M)$.

REMARK. If M is a maximal subfield of the sfield K then $N(M)=N^2(M)$. For by the maximality M is the centralizer of its center.

THEOREM 1. *If F is a proper subfield of the sfield K and if F' is subinvariant in K' then F is in the center of K.*

PROOF. Suppose F is not in the center of K and suppose that $F' \triangle G_1 \triangle G_2 \triangle \cdots \triangle G_n = K'$. We shall show that the sfield $\overline{F}$ generated by all the conjugates of F' in K' is Abelian. This will give a contradiction to the Cartan-Brauer-Hua theorem since then $\overline{F}'$ is invariant in K' but not equal to K' nor in the center of K'.

$\overline{F}$ is not in the center of K since F is not, and $\overline{F}$ contains F. $\overline{F}$ is not equal to K since that would imply K is Abelian and F would be in the center of K. $\overline{F}$ is invariant in K since it is the sfield generated by an invariant subset of K'. Thus the theorem is proved when we show that $\overline{F}$ is Abelian. This will be done by induction on the length n of the composition series containing F'.

Suppose then that for j in some set J, F_j' are all the conjugates of

F' by elements of G_2. Then each F_j' is normal in G_1 and hence by Lemma 2 each F_j' is in the centralizer of all the others. It follows that the sfield F_1 generated by the F_j is a field. Suppose now that we have shown that the sfield F_m generated by all the conjugates of F' in G_m is a field. It is easy to check that F_m' is normal in the group generated by F_m' and G_{m+1}. If $F^{*\prime}$ is a conjugate of F' in G_{m+1} then again by Lemma 2 F^* is in the centralizer of F_m and hence in particular F and F^* commute elementwise; by a symmetry argument all the conjugates of F' contained in G_{m+1} commute elementwise and hence the sfield F_{m+1} that they generate is a field. Then by induction we see that $F_n = \overline{F}$ is Abelian as was to be shown. This proves the theorem.

LEMMA 3. *If M is a noncommutative subsfield of K and if $N(M) \neq N^2(M)$ then $N(M)$ is of index 2 in $N^2(M)$ and $Z'(M)$ is the only other conjugate of M' contained in $N(M)$. Furthermore $N^2(M)$ is its own normalizer in K' and $N^2(M) \neq K'$ provided that the center of M contains at least 5 elements of the center of K.*

PROOF. Suppose $N(M) \neq N^2(M)$ and that $N(M)$ is of index $m > 2$ in $N^2(M)$. Then there are at least three conjugates M', $M^{*\prime}$ and $M^{**\prime}$ contained in $N(M)$ and having $N(M)$ for normalizer. It follows from Lemma 2 that any two of these are in the centralizer of the third and since $N(M)$ is the normalizer of each, the sfield generated by each pair is in $N(M)$. Now since M is not commutative there is an x in M, x not in $Z(M)$. Since M and M^* are distinct conjugates there is a y in M^*, not in M. Then $x+y$ is not in M nor in $Z(M)$. But this contradicts Lemma 1 since both $x+y$ and $x+y+1$ are in $N(M)$ since they are in the centralizer of M^*. We conclude that if $N(M) \neq N^2(M)$ then $N(M)$ is of index 2 in $N^2(M)$.

Now when $N(M)$ is of index 2 in $N^2(M)$ then there is at least one conjugate $M^{*\prime}$ of M' in $N(M)$. M^* is contained in $Z(M)$ and in fact is equal to $Z(M)$; for if M^* were properly contained in $Z(M)$ then by symmetry M would be properly contained in a sfield H such that H' is in $N(M)$. But then by Lemma 2, H would be contained in $Z(M)$, whence M would be also and hence M would be Abelian contrary to hypothesis. We conclude that M^* must be equal to $Z(M)$.

Now if $M^{**\prime}$ were another conjugate of M' in $N(M)$ then M^{**} would be contained in $Z(M) = M^*$ which contradicts the fact that one conjugate cannot be contained in another. We conclude that there are only two conjugates of M' in $N(M)$ when the index of $N(M)$ in $N^2(M)$ is 2 as the lemma asserts.

The following Lemma is now needed to finish the proof of Lemma 3.

LEMMA 4. *There are no noncommutative subsfields L and M of K such that $L \cap N(M)$ is of index 2 in L' provided that the center of M contains at least 5 elements of the center of K.*

PROOF OF LEMMA 4. Let L^* denote $L \cap N(M)$. We shall show first that every x in L^* is either in M or in $Z(M)$. For suppose there is an x in L^* but not in M nor in $Z(M)$. Then by Lemma 1, $x+1$ and $x-1$ are not in L^* and since the index of L^* in L' is 2 it follows that $(x+1)(x-1)=x^2-1$ must be in L^* as is also x^2. It follows again from Lemma 1 that x^2 must be in M or in $Z(M)$.

Now if the characteristic of the sfield is not 2 or 3, let $a=1, b=3, c=2$, and $d=-1$. If the characteristic is 3 let $a=c=1$ and let b and d be distinct elements of M in the center of K but not 0, 1, or 2. If the characteristic is 2 let a, b, c, d be elements of M in the center of K but not 0 or 1 and such that $a+b\neq 0$, $a+b+1\neq 0$ and $a=c$, $d=b+1$. Then none of the elements $x+a$, $x+b$, $x+c$, $x+d$, $x+a+1$, $x+b-1$, $x+c+1$, $x+d-1$ is in L^* so that $(x+a)(x+b)=x^2+(a+b)x+ab$ is in L^* as is also $(x+a+1)(x+b-1)=x^2+(a+b)x+(a+1)\cdot(b-1)$. It follows again from Lemma 1 that $x^2+(a+b)x+ab$ and hence $x^2+(a+b)x$ is in M or in $Z(M)$. Similarly by using c in place of a, d in place of b we see that $x^2+(c+d)x$ is in M or in $Z(M)$. But then two of the three elements x^2, $x^2+(a+b)x$, $x^2+(c+d)x$ are in the same sfield M or $Z(M)$ and by subtraction of one from the other we see that x is also there contrary to the supposition that x was neither in M nor in $Z(M)$. We conclude that every element of L^* is in M or in $Z(M)$.

Now if L^* were in M or if L^* were in $Z(M)$ then that sfield contains all the squares of elements of L since L^* is of index 2 in L' and hence contains L itself since by Theorem 5 of [2] the square elements of a noncommutative sfield generate the whole sfield. This, of course, means that L is contained in M and hence $N(M)$ contrary to the fact that $L \cap N(M)$ is of index 2 in L.

On the other hand, if there are elements x and y of L^*, x in M but not in $Z(M)$ and y in $Z(M)$ but not in M then xy is in L^* but in neither M nor $Z(M)$ contrary to what was shown above. This proves Lemma 4.

We now continue the proof of Lemma 3. If $N(M)\neq N^2(M)$ and if $N^2(M)$ is not its own normalizer then there is a conjugate N^* of $N(M)$ also of index 2 in $N^2(M)$ and in N^* a conjugate $M^{**\prime}$ of M' such that either $M^{**\prime}\cap N(M)$ is of index 1 or 2 in $M^{**\prime}$. We rule out the possibility of this index being 2 because of Lemma 4, while if the index is 1 then $M^{**\prime}$ is contained in $N(M)$ and there are three distinct conjugates of M' in $N(M)$ contrary to the first statement of the lemma already proved. This concludes the proof of Lemma 3.

THEOREM 2. *If M is a proper noncommutative subsfield of a sfield K containing at least* 5 *elements of the center of K, then M' is not subinvariant in K'.*

PROOF. Suppose $M' \triangle G_1 \triangle G_2 \triangle \cdots \triangle G_n = K'$ and suppose r is the largest integer so that G_r is contained in $N^2(M)$. Then $r \neq n$ since $N^2(M) \neq K'$ by Lemma 3. Now if y is any element of G_{r+1} then y transforms M' into a conjugate $M^{*\prime}$ contained in G_r and hence in $N^2(M)$. It follows from Lemmas 3 and 4 that M^* is either M or $Z(M)$ and hence y is in $N^2(M)$; consequently G_{r+1} is also in $N^2(M)$ contrary to the choice of r. We conclude that M' cannot be subinvariant in K'.

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