ON A THEOREM OF PUTNAM AND WINTNER

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In a recent paper [1] Putnam and Wintner prove the following theorem: *Let* a, b *be* $n \times n$ *matrices over a field of characteristic* 0 *such that* a *commutes with* $ab-ba$; *suppose further that* a *and* b *are regular, then* $ab^{-1}a^{-1}b-1$ *is nilpotent.* In this paper they also obtain the analogous result for a, b in a Banach algebra provided that a has a logarithm which commutes with it.

In their proof they make use of the exponentiation of a matrix. We give here a proof for their result which is purely algebraic and which holds true for matrices over any field provided the characteristic of the field is large enough. We also point out that certain information can be obtained from our approach in the case of Banach algebras.

We prove the

THEOREM. *Let* a, b *be regular* $n \times n$ *matrices over a field* F *of characteristic* $p > n$; *suppose that* a *commutes with* $ab-ba$. *Then* $ab^{-1}a^{-1}b-1$ *is nilpotent.*

PROOF. In order to prove the theorem it is clearly sufficient to show that 1 is the only characteristic root of $ab^{-1}a^{-1}b$.

So suppose that λ is a characteristic root of $ab^{-1}a^{-1}b$. Since a, b are regular $\lambda \neq 0$. Thus $ab^{-1}a^{-1}b-\lambda$ fails to have an inverse. But then, by multiplying through by ba^{-1} from the left, $a^{-1}b-\lambda ba^{-1}$ does not have an inverse, that is, $\lambda(a^{-1}b-ba^{-1})+(1-\lambda)a^{-1}b$ does not have an inverse. Since $a^{-1}b-ba^{-1}=a^{-1}(ba-ab)a^{-1}=a^{-2}(ba-ab)$, by our hypothesis, we now have that $\lambda a^{-2}(ba-ab)+(1-\lambda)a^{-1}b$ is singular. Thus if we multiply this on the left by a^2 the resulting element, $\lambda(ba-ab)+(1-\lambda)ab=\lambda ba+(1-2\lambda)ab$ is also not invertible. Hence, multiplying through by $b^{-1}a^{-1}$ from the left, $b^{-1}a^{-1}ba-((2\lambda-1)/\lambda)$ fails to have an inverse. Consequently $(2\lambda-1)/\lambda$ is a characteristic root of $b^{-1}a^{-1}ba$. Since $b^{-1}a^{-1}ba=a^{-1}(ab^{-1}a^{-1}b)a$, these matrices have the same characteristic roots and so we have that $(2\lambda-1)/\lambda$ is a characteristic root of $ab^{-1}a^{-1}b$. *Thus if* λ *is a characteristic root of* $ab^{-1}a^{-1}b$ *then so is* $(2\lambda-1)/\lambda$. Iterating this we have that for all integers k, $(k\lambda-(k-1))/((k-1)\lambda-(k-2))$ is a characteristic root of $ab^{-1}a^{-1}b$. Since $ab^{-1}a^{-1}b$ has at most n roots, we have that for $0 \leq k < k' \leq n$,

Received by the editors October 4, 1957.

$$\frac{k\lambda-(k-1)}{(k-1)\lambda-(k-2)}=\frac{k'\lambda-(k'-1)}{(k'-1)\lambda-(k'-2)}$$

the net result of which is $(k-k')(\lambda-1)^2=0$. Since $0\leqq k<k'\leqq n$, and the characteristic p of F is larger than n, $k-k'\not\equiv 0(p)$ and so $(\lambda-1)^2=0$ results. Hence $\lambda=1$ and the theorem is proved.

We point out that the argument used above also works, up to a point in a Banach algebra. The conclusion one can reach is that the spectrum of $ab^{-1}a^{-1}b$ is invariant under the transformation $(2\lambda-1)/\lambda$.

Bibliography

1. C. Putnam and A. Wintner, *On the spectra of commutators*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 360–362.

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SOME ORTHOGONAL FUNCTIONS CONNECTED WITH POLYNOMIAL IDENTITIES

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If $P(x)$ is an arbitrary polynomial of sufficiently small degree then there are various ways of choosing an integer N and coefficients c_j such that

$$\sum_{j=0}^{N} c_jP(x+j)=0 \tag{1}$$

for all x. In each of [**3**; **4**] such an identity is proved. The present paper is devoted to a discussion of a connection between the coefficients in these identities and certain classes of orthonormal functions.

1. A polynomial identity and the Walsh functions. In [4] we proved the identity

$$\sum_{n=0}^{b_1^{k+1}-1} \omega^{v_b(n)}P(x+n)=0, \tag{2}$$

where b and k are positive integers, $v_b(n)$ is the sum of the coefficients

Presented to the Society, August 30, 1957 under the title *Some classes of orthogonal functions;* received by the editors December 16, 1957.