

ON THE ARITHMETIC AND GEOMETRIC MEANS AND ON HÖLDER'S INEQUALITY

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1. The theorem on the arithmetic and geometric means can be sharpened; in consequence of this, Hölder's inequality can be sharpened also, and finally, Minkowski's inequality.

It is well known that the difference between the two means,

$$(1.1) \quad D_n = (x_1 + x_2 + \cdots + x_n)/n - (x_1 x_2 \cdots x_n)^{1/n},$$

is the square of an irrational function for $n=2$, a sum of squares for $n>2$. For instance¹

$$D_3 = \sum_{1 \leq i < k \leq 3} \left\{ (x_1^{1/3} + x_2^{1/3} + x_3^{1/3})^{1/2} (x_i^{1/3} - x_k^{1/3}) / 6^{1/2} \right\}^2.$$

These representations are increasingly complicated with increasing n . But is there not a *simple* sum of squares that is comparable² with D_n in the Hardy-Littlewood-Pólya sense? In fact, $\sum ((x_i)^{1/2} - (x_k)^{1/2})^2$ is, as will be shown.

Throughout this paper we suppose that *not all the x_i be equal*; and we write $\sum u_{ik}$ or $\sum^n u_{ik}$ when $1 \leq i < k \leq n$. We prove

THEOREM 1. *Let $n \geq 2$, $0 < q_1 \leq q_2 \leq \cdots \leq q_n$, $\sum q_i = 1$, $x_i \geq 0$. Then*

$$(1.2) \quad q_1(n-1)^{-1} \leq \frac{\Delta_n}{\sum ((x_i)^{1/2} - (x_k)^{1/2})^2} \leq q_n;$$

$$\Delta_n = \sum_1^n q_i x_i - x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}.$$

If $q_1 = q_2 = \cdots = q_m$ for some m , $1 \leq m \leq n$, then the lower bound $q_1(n-1)^{-1}$ is reached if and only if any one of the x_i with $1 \leq i \leq m$ is positive and the other $n-1$ numbers x_i vanish; except for the trivial case $n=2$, $q_1 = q_2 = 1/2$. If $q_1 < q_2 = q_3 = \cdots = q_n$ then the upper bound q_n is attained if and only if $x_1 = 0$ and $x_2 = x_3 = \cdots = x_n > 0$. If $q_1 = q_2 = \cdots = q_n = 1/n$ and $n \geq 3$ then the upper bound $1/n$ is

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¹ G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge, 1934, see §2.23.

² Loc. cit., §§1.6, 3.4, where one-sided comparability is introduced. Here double-sided comparability of two functions f and g is considered, i.e. two inequalities $0 \leq f \leq c_1 g$ and $0 \leq g \leq c_2 f$, where c_1 and c_2 are positive constants.

attained if and only if any one of the x_i is zero and the other $n - 1$ numbers x_i are positive and equal.

REMARKS. For (b), cf. Appendix.

(a) It is an open question whether q_n is the *least* upper bound *in the general case*.

(b) When the point $\{x_1, x_2, \dots, x_n\}$ in n -dimensional space ($n > 2$) tends to $\{x_0, x_0, \dots, x_0\}$, where $x_0 > 0$, the fraction in (1.2) tends to a unique limit if, and only if, $q_1 = q_2 = \dots = q_n = 1/n$. This limit is $2n^{-2}$.

Concerning Hölder's inequality we deduce

THEOREM 2. *Let $0 < q_1 \leq q_2 \leq \dots \leq q_n$, $\sum q_i = 1$ and $a_{ik} \geq 0$; $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$; $m, n \geq 2$; $a_{1k} + a_{2k} + \dots + a_{mk} = s_k > 0$ for $k = 1, 2, \dots, n$. Then (cf. 5.1)*

$$\sum_{i=1}^m a_{i1}^{q_1} a_{i2}^{q_2} \dots a_{in}^{q_n} \geq \max \{0; s_1^{q_1} s_2^{q_2} \dots s_n^{q_n} (1 - q_n L)\},$$

(1.3)

$$\sum_{i=1}^m a_{i1}^{q_1} a_{i2}^{q_2} \dots a_{in}^{q_n} \leq s_1^{q_1} s_2^{q_2} \dots s_n^{q_n} \left(1 - \frac{q_1}{n-1} L\right),$$

where

$$L = \sum_{i=1}^m \sum_{1 \leq j < k \leq n} \left\{ \left(\frac{a_{ij}}{s_j}\right)^{1/2} - \left(\frac{a_{ik}}{s_k}\right)^{1/2} \right\}^2.$$

If $n = 2$ and $q_1 = q_2 = 1/2$, then trivially $\sum (a_{i1} a_{i2})^{1/2} = (s_1 s_2)^{1/2} (1 - L/2)$.

The corresponding problem for Minkowski's inequality is more difficult, it plays a part in an investigation into arc length and surface area. One result is given in §6.

2. On the lower bound in Theorem 1. First take $q_1 < q_2$ and set $\Delta_n - \sum ((x_i)^{1/2} - (x_k)^{1/2})^2 q_1 (n-1)^{-1} = d_n$. Then

$$d_n = \sum_{i=2}^n (q_i - q_1) x_i + \{2q_1 (n-1)^{-1}\} \sum^n (x_i x_k)^{1/2} - x_1^{q_1} x_2^{q_2} \dots x_n^{q_n}.$$

But

$$\sum (q_i - q_1) + \frac{n(n-1)}{2} 2q_1 (n-1)^{-1} = (1 - nq_1) + nq_1 = 1.$$

Using the Theorem on the arithmetic and geometric means in its

generalized form³ we deduce that $d_n > 0$, unless

$x_2 = x_3 = \dots = x_n = (x_1x_2)^{1/2} = (x_1x_3)^{1/2} = \dots = (x_2x_3)^{1/2} = \dots$,
 i.e. unless $x_1 > 0$ and $x_2 = x_3 = \dots = x_n = 0$, since not all the x_i are equal. In a similar way we deal with the cases $q_1 = q_2 < q_3$, $q_1 = q_2 = q_3 < q_4$, etc.

3. Now we need a lemma.

If $a_i \geq 0$, $n > 2$, not all the a_i are equal and

$$(3.1) \quad g_n(a) = (n - 2) \sum_{i=1}^n a_i + n(a_1a_2 \cdots a_n)^{1/n} - 2 \sum_{i=1}^n (a_i a_k)^{1/2},$$

then $g_n(a) > 0$, unless one of the a_i is zero and the others are positive and equal.

Without loss of generality we assume that $a_1 \leq a_2 \leq \dots \leq a_n$. Trivially $g_2(a) \equiv 0$. Suppose it be true that, for some $n > 2$, $g_n(a) \geq 0$ for all (a_1, a_2, \dots, a_n) , $a_i \geq 0$. We consider $g_{n+1}(a)$ and set $a_1^* = a_1$, $a_2^* = a_3^* = \dots = a_{n+1}^* = A = (a_2a_3 \cdots a_{n+1})^{1/n}$. Then

$$g_{n+1}(a^*) = (n - 1)a_1 + (n + 1)a_1^{1/(n+1)} A^{n/(n+1)} - 2n(a_1A)^{1/2}.$$

If $a_1 = 0$ then $g_{n+1}(a^*) = 0$ also. Let now $a_1 > 0$. Then

$$\frac{g_{n+1}(a^*)}{2na_1^{1/(n+1)}} = q_1 a_1^{n/(n+1)} + q_2 A^{n/(n+1)} - a_1^{q_1 n/(n+1)} A^{q_2 n/(n+1)},$$

where $q_1 = (n - 1)/(2n)$, $q_2 = (n + 1)/(2n)$; $q_1 + q_2 = 1$. Hence $g_{n+1}(a^*) > 0$, unless $a_1 = A$, i.e. $a_1 = a_2 = \dots = a_{n+1}$; which case, however, was excluded. Thus $g_{n+1}(a^*) > 0$ unless $a_1 = 0$.

Now we consider $g_{n+1}(a) - g_{n+1}(a^*) = d_{n+1}$. We have

$$\begin{aligned} d_{n+1} &= (n - 1) \sum_{i=2}^{n+1} a_i - 2(a_1)^{1/2} \left(\sum_{i=2}^{n+1} (a_i)^{1/2} - n(A)^{1/2} \right) \\ &\quad - 2 \sum_{2 \leq i < k \leq n+1} (a_i a_k)^{1/2} \\ &= g_n(a_2, a_3, \dots, a_{n+1}) + \sum_{i=2}^{n+1} a_i \\ &\quad - 2(a_1)^{1/2} \left(\sum_{i=2}^{n+1} (a_i)^{1/2} - n(A)^{1/2} \right) - n \prod_{i=2}^{n+1} a_i^{1/n}. \end{aligned}$$

Since $g_n(a_2, a_3, \dots, a_{n+1}) \geq 0$ by assumption, $a_1 \leq A$ and $(a_2a_3 \cdots a_{n+1})^{1/n} = A$,

³ Loc. cit., Theorem 9, Inequality (2.5.2).

$$d_{n+1} \geq \sum_{i=2}^{n+1} a_i - 2(A)^{1/2} \sum_{i=2}^{n+1} (a_i)^{1/2} + nA = \sum_{i=2}^{n+1} ((a_i)^{1/2} - (A)^{1/2})^2.$$

This is positive, unless $a_i = A$ ($i = 2, 3, \dots, n+1$), i.e. $a_2 = a_3 = \dots = a_{n+1}$. Combining the two results we complete the proof.

4. On the upper bound in (1.2). The denominator is

$$(4.1) \quad \begin{aligned} & \sum ((x_i)^{1/2} - (x_k)^{1/2})^2 \\ & = nD_n + \{(n-2) \sum x_i + n(x_1x_2 \dots x_n)^{1/n} - 2 \sum (x_ix_k)^{1/2}\}. \end{aligned}$$

By the lemma, the term in the curled brackets is positive, unless one of the x_i is zero and the others are equal. Under this restriction, therefore, the fraction in (1.2) is smaller than $\Delta_n/(nD_n)^{-1}$.

(i) When $q_1 = q_2 = \dots = q_n = 1/n$ clearly $\Delta_n = D_n$, which completes the proof for the familiar case,

(ii) If not all the q_i are equal, not all the differences $q_n - q_i$ vanish. Since

$$(4.2) \quad D_n - \frac{\Delta_n}{nq_n} = \frac{1}{nq_n} \sum_{i=1}^{n-1} (q_n - q_i)x_i + \frac{1}{nq_n} \prod_{i=1}^n x_i^{q_i} - \prod_{i=1}^n x_i^{1/n}$$

and

$$\sum_{i=1}^{n-1} \frac{q_n - q_i}{nq_n} + \frac{1}{nq_n} = 1; \quad \frac{q_n - q_i}{nq_n} + \frac{q_i}{nq_n} = \frac{1}{n},$$

we can use the Theorem on the means. Hence $D_n - \Delta_n/(nq_n) \geq 0$, therefore $\Delta_n/(nD_n) \leq q_n$. Thus

$$(4.3) \quad \Delta_n / \sum ((x_i)^{1/2} - (x_k)^{1/2})^2 \leq q_n.$$

If $q_2 = q_3 = \dots = q_n > q_1$ the first term on the right in (4.2) reduces to $(q_n - q_1)(nq_n)^{-1}x_1$; therefore $D_n - \Delta_n/(nq_n) = 0$ if, and only if, $x_1 = x_1^{q_1}x_2^{q_2} \dots x_n^{q_n}$. Combining this with the above condition resulting from the lemma we deduce that $x_1 = 0, x_2 = x_3 = \dots = x_n > 0$, which completes the proof.

5. On Hölder's inequality. We employ a well-known argument.⁴ Set $a_{ik}/s_k = b_{ik}$; then $b_{1k} + b_{2k} + \dots + b_{mk} = 1$ for each k . Now

$$1 - \sum_{i=1}^m b_{i1}^{q_1} b_{i2}^{q_2} \dots b_{in}^{q_n} = \sum_{i=1}^m \left(q_1 b_{i1} + q_2 b_{i2} + \dots - q_n b_{in} + \prod_{k=1}^n b_{ik}^{q_k} \right).$$

We can apply Theorem 1 to each of the expressions in brackets. Thus

⁴ Loc. cit. §2.7, proof (ii) of Hölder's inequality.

$$(5.1) \quad \frac{q_1}{n-1} \sum_{i=1}^m \sum_{1 \leq j < k \leq n} ((b_{ij})^{1/2} - (b_{ik})^{1/2})^2 \leq 1 - \sum_{i=1}^m \prod_{k=1}^n b_{ik}^{q_k} \leq \min \left\{ 1, q_n \sum_{i=1}^m \sum_{1 \leq j < k \leq n} ((b_{ij})^{1/2} - (b_{ik})^{1/2})^2 \right\},$$

which is equivalent to the result required. Clearly it implies that $\sum b_{i1}^{q_1} b_{i2}^{q_2} \cdots b_{in}^{q_n} = 1$ if, and only if, for each i, j and k the numbers b_{ij} and b_{ik} are equal.

Henceforward we exclude this. In special cases certainly equality in (5.1) can be attained. For $n=2, m \geq 2$, see §1. When $m=n$ and $q_1=q_2=\cdots=q_n=1/n$, for instance, there is equality on the left if and only if, for some permutation $\kappa, \lambda, \dots, \omega$ of the numbers $1, 2, \dots, n, b_{1\kappa}=b_{2\lambda}=\cdots=b_{m\omega}=1$; equality on the right if and only if $b_{1\kappa}=b_{2\lambda}=\cdots=b_{m\omega}=0$ while all the other b_{ik} are positive and equal (i.e., $=1/(m-1)$).

6. A general, yet preliminary, result on Minkowski's inequality.

THEOREM 3. *Let $m \geq 2, n \geq 2, r > 1, r' = r(r-1)^{-1}, a_{ik} \geq 0, \rho_k^r = \sum_{i=1}^m a_{ik}^r > 0,$*

$$S^r = \sum_{i=1}^m \left(\sum_{k=1}^n a_{ik} \right)^r,$$

$$L_k = \sum_{i=1}^m \left\{ \left(\frac{a_{i1} + \cdots + a_{in}}{S} \right)^{r/2} - \left(\frac{a_{ik}}{\rho_k} \right)^{r/2} \right\}^2.$$

Then

$$(6.1) \quad \frac{1}{\max(r, r')} \sum_{k=1}^n \rho_k L_k \leq \sum_{k=1}^n \rho_k - S \leq \frac{1}{\min(r, r')} \sum_{k=1}^n \rho_k L_k.$$

Clearly there is equality whenever $r=2$; and $\sum_{k=1}^n \rho_k > S$, unless $a_{ik}/\rho_k = a_{il}/\rho_l$ for any i, k, l , i.e. unless the a_{i1}, a_{i2}, \dots are proportional.

The proof is based on a well-known argument.⁵ If $\sigma_i = \sum_{k=1}^n a_{ik}$, $b_{ik} = a_{ik}^r$, then

$$(6.2) \quad S = S^{1-r} \left\{ \sum_{i=1}^m a_{i1} \sigma_i^{r-1} + \sum_{i=1}^m a_{i2} \sigma_i^{r-1} + \cdots + \sum_{i=1}^m a_{in} \sigma_i^{r-1} \right\},$$

$$(6.3) \quad a_{ik} \sigma_i^{r-1} = b_{ik}^{1/r} (\sigma_i^r)^{1/r'}.$$

We fix k ($= 1, \text{ or } 2, \dots, n$) and use Theorem 2. If $r > r'$ let $q_1 = 1/r$,

⁵ Loc. cit., §2.11, first proof of Minkowski's inequality.

$q_2 = 1/r'$; $s_1 = \sum_{i=1}^m b_{ik} = \rho_k^r$, $s_2 = \sum_{i=1}^m \sigma_i^r = S^r$; we deduce that

$$(6.4) \quad \rho_k S^{r-1} (1 - q_2 L_k) \leq \sum_{i=1}^m b_{ik}^{1/r} (\sigma_i^r)^{1/r'} \leq \rho_k S^{r-1} (1 - q_1 L_k).$$

If $r < r'$ interchange q_1 and q_2 , and s_1 and s_2 also. Combining these results we can complete the proof. A theorem which is more suitable for applications will be given in another paper.

Appendix. Added on October 10, 1957. We prove the Remark (b) to Theorem 1; and in addition, two results: if $n=2$, $q_1 \geq q_2$, the fraction in (1.2) tends to a unique limit when $(x_1 - x_0)^2 + (x_2 - x_0)^2 \rightarrow 0$, where $x_0 > 0$ is fixed; for $n \geq 2$, there is never a unique limit when $x_0 = 0$, except for the trivial case $n=2$, $q_1 = q_2 = 1/2$.

We set $U_n(x) = \sum_{i=1}^n ((x_i)^{1/2} - (x_k)^{1/2})^2$ ($1 \leq i < k \leq n$) and take $\sum_i (x_i - x_0)^2 \rightarrow 0$. We observe that not all the x_i are equal.

I. Let $n > 2$, $q_1 < q_n$. If $x_2 = x_3 = \dots = x_n = 0$, $x_1 = x_0 + \epsilon$ ($\epsilon \geq 0$),

$$(1.1) \quad \Delta_n(x) / (U_n(x)) \rightarrow 2(n-1)^{-1} q_1 (1 - q_1) \quad (\epsilon \rightarrow 0)$$

by the binomial series. If, however, $x_1 = \dots = x_{n-1} = 0$, $x_n = x_0 + \epsilon$, the limit is $2(n-1)^{-1} q_n (1 - q_n) > 2(n-1)^{-1} q_1 (1 - q_1)$. Thus there is no unique limit.

II. Let $n > 2$, $q_1 = q_2 = \dots = q_n = 1/n$. We need only show the *existence* of a *unique* limit. Then (1.1) implies that the latter is $2n^{-2}$. Set $x_i = X_i^{2n}$ ($i = 1, 2, \dots, n$), $x_0 = A^{2n}$. Then

$$(1.2) \quad U_n(X^{2n}) = \sum_{1 \leq i < k \leq n} (X_i - X_k)^2 P_{ik}; \quad P_{ik} = \left(\sum_{j=0}^{n-1} X_i^{n-1-j} X_k^j \right)^2.$$

Hence, given a sufficiently small $\epsilon > 0$, there is a δ such that

$$(1.3) \quad 0 < n^2 A^{2n-2} - \epsilon < P_{ik} < n^2 A^{2n-2} + \epsilon, \quad \left\{ \sum_i (X_i - A)^2 < \delta \right\}.$$

Now it follows from the Hurwitz-Muirhead identity⁶ that the polynomial $n \Delta_n(X^{2n}) = \sum_i X_i^{2n} - n X_1^2 X_2^2 \dots X_n^2$ is representable in the form

$$(1.4) \quad n \Delta_n(X^{2n}) = \sum_{1 \leq i < k \leq n} (X_i - X_k)^2 Q_{ik},$$

where the polynomials $Q_{ik} \equiv Q_{ki}$ are of degree $2n-2$ and are homogeneous in the variables X_1, X_2, \dots, X_n , with all their coefficients positive. Let K be the sum of the coefficients of Q_{ik} . Since Q_{ji} may

⁶ Loc. cit., §2.23 (2.23.1) and §2.21(1).

be derived from Q_{ik} by interchanging X_i with X_j and X_k with X_l , K is independent of i, k . Thus, for some $\delta > 0$,

$$(1.5) \quad 0 < KA^{2n-2} - \epsilon < Q_{ik} < KA^{2n-2} + \epsilon, \quad \left\{ \sum_i (X_i - A)^2 < \delta \right\}.$$

By (1.4) and (1.5),

$$\frac{KA^{2n-2} - \epsilon}{n^2 A^{2n-2} + \epsilon} < \frac{n\Delta_n}{U_n} < \frac{KA^{2n-2} + \epsilon}{n^2 A^{2n-2} - \epsilon}.$$

As $\epsilon \rightarrow 0, \Delta_n/U_n \rightarrow Kn^{-3}$; which shows the existence of the limit.

III. The case $n = 2$. We show that the function

$$f(x, y; p) = (px + qy - x^p y^q)/(x^{1/2} - y^{1/2})^2 \quad (0 < p < p + q = 1)$$

tends to $2pq$ as $(x-a)^2 + (y-a)^2 \rightarrow 0$, where $a > 0$ is fixed. If p is rational, this follows readily⁷ from the preceding result.

Let now p be irrational and let $\{P_m\}$ and $\{p_m\}$ ($m = 1, 2, \dots$) be sequences of rational numbers tending to p and such that $P_m > p > p_m$. Then $Q_m \rightarrow q, q_m \rightarrow q$ ($m \rightarrow \infty$), where $Q_m = 1 - P_m, q_m = 1 - p_m$, and

$$(1.6) \quad \lim_{(x,y) \rightarrow (a,a)} f(x, y; P_m) = 2P_m Q_m; \quad \lim_{(x,y) \rightarrow (a,a)} f(x, y; p_m) = 2p_m q_m.$$

(i) Let $x > y > 0, x/y = u (> 1)$, and consider the function $g(s) = sx + (1-s)y - x^s y^{1-s}$ for fixed x, y . Suppose first that $p < 1/2$. We may take $P_m < 1/2$ ($m = 1, 2, \dots$) and $s < 1/2$. Then $2p_m q_m < 2pq < 2P_m Q_m$,

$$\begin{aligned} \frac{g'(s)}{y} &= u - 1 - u^s \log u > u - 1 - u^{1/2} \log u \\ &= u^{1/2} \int_1^u (v^{1/4} - v^{-1/4})^2 \frac{dv}{2v} \\ &> 0. \end{aligned}$$

Hence $g(P_m) > g(p) > g(p_m)$; as $g(p_m) = (x^{1/2} - y^{1/2})^2 f(p_m)$, by (1.6)

$$(1.7) \quad 2p_m q_m \leq \liminf_{(x,y) \rightarrow (a,a)} f(x, y; p) \leq \limsup_{(x,y) \rightarrow (a,a)} f(x, y; p) \leq 2P_m Q_m.$$

Let now $p > 1/2$. Then we may suppose that, for some constant $c > 1/2, c < p_m < p < P_m < 1$ ($m = 1, 2, \dots$). Set $u_0 = \{c(1-c)^{-1}\}^{1/(1-c)}$. Then $u_0 > 1$,

⁷ Loc. cit., cf. §2.2 and the proof of (2.5.1).

$$\begin{aligned} \frac{g'(s)}{yu^c} &\cong \frac{u - 1 - u^c \log u}{u^c} = \int_1^u \{(1-c)v + c - v^c\} \frac{dv}{v^{1+c}} \\ &= c \int_1^u \frac{dv}{v^{1+c}} \int_1^v \left(\frac{1}{u_0^{1-c}} - \frac{1}{z^{1-c}} \right) dz; \end{aligned}$$

which is certainly negative for $1 < u < u_0$. Since $x \rightarrow a$ and $y \rightarrow a$, we may take u , i.e. x/y , $< u_0$. Hence $f(x, y; P_m) < f(x, y; p) < f(x, y; p_m)$, and we obtain an inequality similar to (1.7), with $2p_m q_m$ and $2P_m Q_m$ interchanged. Taking $m \rightarrow \infty$, we have

$$(1.8) \quad \lim_{(x,y) \rightarrow (a,a)} f(x, y; p) = 2pq.$$

(ii) Let now $y > x$. We arrive at the same result when observing that $f(x, y; p) = f(y, x; q) \rightarrow 2qp$; thus we can complete the proof.

IV. The last statement is deduced by an argument similar to that used in I. Thus all the assertions are true.

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