ON THE ARITHMETIC AND GEOMETRIC MEANS AND ON HÖLDER'S INEQUALITY

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1. The theorem on the arithmetic and geometric means can be sharpened; in consequence of this, Hölder's inequality can be sharpened also, and finally, Minkowski's inequality.

It is well known that the difference between the two means,

$$D_n = (x_1 + x_2 + \cdots + x_n)/n - (x_1 x_2 \cdots x_n)^{1/n}, \tag{1.1}$$

is the square of an irrational function for $n=2$, a sum of squares for $n>2$. For instance[1]

$$D_3 = \sum_{1\leq i<k\leq 3} \left\{ \left(x_1^{1/3} + x_2^{1/3} + x_3^{1/3}\right)^{1/2} \left(x_i^{1/3} - x_k^{1/3}\right)/6^{1/2} \right\}^2.$$

These representations are increasingly complicated with increasing n. But is there not a *simple* sum of squares that is comparable[2] with D_n in the Hardy-Littlewood-Pólya sense? In fact, $\sum((x_i)^{1/2}-(x_k)^{1/2})^2$ is, as will be shown.

Throughout this paper we suppose that *not all the x_i be equal*; and we write $\sum u_{ik}$ or $\sum^n u_{ik}$ when $1 \leqq i < k \leqq n$. We prove

THEOREM 1. *Let $n \geqq 2$, $0 < q_1 \leqq q_2 \leqq \cdots \leqq q_n$, $\sum q_i = 1$, $x_i \geqq 0$. Then*

$$(1.2)\qquad \begin{gathered} q_1(n-1)^{-1} \leqq \frac{\Delta_n}{\sum((x_i)^{1/2} - (x_k)^{1/2})^2} \leqq q_n; \\ \Delta_n = \sum_1^n q_i x_i - x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}. \end{gathered}$$

If $q_1 = q_2 = \cdots = q_m$ for some m, $1 \leqq m \leqq n$, then the lower bound $q_1(n-1)^{-1}$ is reached if and only if any one of the x_i with $1 \leqq i \leqq m$ is positive and the other $n-1$ numbers x_i vanish; except for the trivial case $n=2$, $q_1=q_2=1/2$. If $q_1 < q_2 = q_3 = \cdots = q_n$ then the upper bound q_n is attained if and only if $x_1 = 0$ and $x_2 = x_3 = \cdots = x_n > 0$. If $q_1 = q_2 = \cdots = q_n = 1/n$ and $n \geqq 3$ then the upper bound $1/n$ is

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[1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge, 1934, see §2.23.

[2] Loc. cit., §§1.6, 3.4, where one-sided comparability is introduced. Here double-sided comparability of two functions f and g is considered, i.e. two inequalities $0 \leqq f \leqq c_1 g$ and $0 \leqq g \leqq c_2 f$, where c_1 and c_2 are positive constants.

attained if and only if any one of the x_i is zero and the other $n-1$ numbers x_i are positive and equal.

REMARKS. For (b), cf. Appendix.

(a) It is an open question whether q_n is the *least* upper bound *in the general case.*

(b) When the point $\{x_1, x_2, \cdots, x_n\}$ in n-dimensional space $(n>2)$ tends to $\{x_0, x_0, \cdots, x_0\}$, where $x_0>0$, the fraction in (1.2) tends to a unique limit if, and only if, $q_1=q_2=\cdots=q_n=1/n$. This limit is $2n^{-2}$.

Concerning Hölder's inequality we deduce

THEOREM 2. *Let* $0<q_1\leqq q_2\leqq\cdots\leqq q_n$, $\sum q_i=1$ *and* $a_{ik}\geqq 0$; $i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$; $m, n\geqq 2$; $a_{1k}+a_{2k}+\cdots+a_{mk}=s_k>0$ *for* $k=1, 2, \cdots, n$. *Then* (*cf.* 5.1)

$$\sum_{i=1}^{m} a_{i1}^{q_1}a_{i2}^{q_2}\cdots a_{in}^{q_n} \geqq \max\left\{0; s_1^{q_1}s_2^{q_2}\cdots s_n^{q_n}(1-q_nL)\right\},$$

$$\tag{1.3} \sum_{i=1}^{m} a_{i1}^{q_1}a_{i2}^{q_2}\cdots a_{in}^{q_n} \leqq s_1^{q_1}s_2^{q_2}\cdots s_n^{q_n}\left(1-\frac{q_1}{n-1}L\right),$$

where

$$L=\sum_{i=1}^{m}\sum_{1\leq j<k\leq n}\left\{\left(\frac{a_{ij}}{s_j}\right)^{1/2}-\left(\frac{a_{ik}}{s_k}\right)^{1/2}\right\}^2.$$

If $n=2$ and $q_1=q_2=1/2$, then trivially $\sum(a_{i1}a_{i2})^{1/2}=(s_1s_2)^{1/2}(1-L/2)$.

The corresponding problem for Minkowski's inequality is more difficult, it plays a part in an investigation into arc length and surface area. One result is given in §6.

2. On the lower bound in Theorem 1. First take $q_1<q_2$ and set $\Delta_n-\sum((x_i)^{1/2}-(x_k)^{1/2})^2q_1(n-1)^{-1}=d_n$. Then

$$d_n=\sum_{i=2}^{n}(q_i-q_1)x_i+\left\{2q_1(n-1)^{-1}\right\}\sum(x_ix_k)^{1/2}-x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n}.$$

But

$$\sum(q_i-q_1)+\frac{n(n-1)}{2}2q_1(n-1)^{-1}=(1-nq_1)+nq_1=1.$$

Using the Theorem on the arithmetic and geometric means in its generalized form[3] we deduce that $d_n > 0$, unless

$$x_2 = x_3 = \cdots = x_n = (x_1x_2)^{1/2} = (x_1x_3)^{1/2} = \cdots = (x_2x_3)^{1/2} = \cdots,$$

i.e. unless $x_1 > 0$ and $x_2 = x_3 = \cdots = x_n = 0$, since not all the x_i are equal. In a similar way we deal with the cases $q_1 = q_2 < q_3$, $q_1 = q_2 = q_3 < q_4$, etc.

3. **Now we need a lemma.**

If $a_i \geqq 0$, $n > 2$, not all the a_i are equal and

$$(3.1) \quad g_n(a) = (n-2)\sum_{i=1}^{n} a_i + n(a_1a_2\cdots a_n)^{1/n} - 2\sum^{n}(a_ia_k)^{1/2},$$

then $g_n(a) > 0$, unless one of the a_i is zero and the others are positive and equal.

Without loss of generality we assume that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$. Trivially $g_2(a) \equiv 0$. Suppose it be true that, for some $n > 2$, $g_n(a) \geqq 0$ for all $(a_1, a_2, \cdots, a_n)$, $a_i \geqq 0$. We consider $g_{n+1}(a)$ and set $a_1^* = a_1$, $a_2^* = a_3^* = \cdots = a_{n+1}^* = A = (a_2a_3\cdots a_{n+1})^{1/n}$. Then

$$g_{n+1}(a^*) = (n-1)a_1 + (n+1)a_1^{1/(n+1)}A^{n/(n+1)} - 2n(a_1A)^{1/2}.$$

If $a_1 = 0$ then $g_{n+1}(a^*) = 0$ also. Let now $a_1 > 0$. Then

$$\frac{g_{n+1}(a^*)}{2na_1^{1/(n+1)}} = q_1a_1^{n/(n+1)} + q_2A^{n/(n+1)} - a_1^{q_1n/(n+1)}A^{q_2n/(n+1)},$$

where $q_1 = (n-1)/(2n)$, $q_2 = (n+1)/(2n)$; $q_1 + q_2 = 1$. Hence $g_{n+1}(a^*) > 0$, unless $a_1 = A$, i.e. $a_1 = a_2 = \cdots = a_{n+1}$; which case, however, was excluded. Thus $g_{n+1}(a^*) > 0$ unless $a_1 = 0$.

Now we consider $g_{n+1}(a) - g_{n+1}(a^*) = d_{n+1}$. We have

$$\begin{aligned} d_{n+1} &= (n-1)\sum_{i=2}^{n+1} a_i - 2(a_1)^{1/2}\left(\sum_{i=2}^{n+1}(a_i)^{1/2} - n(A)^{1/2}\right) \\ &\qquad -2\sum_{2\leq i<k\leq n+1}(a_ia_k)^{1/2} \\ &= g_n(a_2, a_3, \cdots, a_{n+1}) + \sum_{i=2}^{n+1} a_i \\ &\qquad - 2(a_1)^{1/2}\left(\sum_{i=2}^{n+1}(a_i)^{1/2} - n(A)^{1/2}\right) - n\prod_{i=2}^{n+1} a_i^{1/n}. \end{aligned}$$

Since $g_n(a_2, a_3, \cdots, a_{n+1}) \geqq 0$ by assumption, $a_1 \leqq A$ and $(a_2a_3\cdots a_{n+1})^{1/n} = A$,

[3] Loc. cit., Theorem 9, Inequality (2.5.2).

$$d_{n+1} \geqq \sum_{i=2}^{n+1} a_i - 2(A)^{1/2} \sum_{i=2}^{n+1} (a_i)^{1/2} + nA = \sum_{i=2}^{n+1} ((a_i)^{1/2} - (A)^{1/2})^2.$$

This is positive, unless $a_i = A$ $(i = 2, 3, \cdots, n+1)$, i.e. $a_2 = a_3 = \cdots = a_{n+1}$. Combining the two results we complete the proof.

4. On the upper bound in (1.2). The denominator is

$$\begin{aligned} &\sum((x_i)^{1/2} - (x_k)^{1/2})^2 \\ &= nD_n + \left\{(n-2)\sum x_i + n(x_1x_2 \cdots x_n)^{1/n} - 2\sum (x_ix_k)^{1/2}\right\}. \end{aligned} \tag{4.1}$$

By the lemma, the term in the curled brackets is positive, unless one of the x_i is zero and the others are equal. Under this restriction, therefore, the fraction in (1.2) is smaller than $\Delta_n(nD_n)^{-1}$.

(i) When $q_1 = q_2 = \cdots = q_n = 1/n$ clearly $\Delta_n = D_n$, which completes the proof for the familiar case,

(ii) If not all the q_i are equal, not all the differences $q_n - q_i$ vanish. Since

$$D_n - \frac{\Delta_n}{nq_n} = \frac{1}{nq_n} \sum_{i=1}^{n-1} (q_n - q_i)x_i + \frac{1}{nq_n} \prod_{i=1}^{n} x_i^{q_i} - \prod_{i=1}^{n} x_i^{1/n} \tag{4.2}$$

and

$$\sum_{i=1}^{n-1} \frac{q_n - q_i}{nq_n} + \frac{1}{nq_n} = 1; \qquad \frac{q_n - q_i}{nq_n} + \frac{q_i}{nq_n} = \frac{1}{n},$$

we can use the Theorem on the means. Hence $D_n - \Delta_n/(nq_n) \geqq 0$, therefore $\Delta_n/(nD_n) \leqq q_n$. Thus

$$\Delta_n \Big/ \sum^{n} ((x_i)^{1/2} - (x_k)^{1/2})^2 \leqq q_n. \tag{4.3}$$

If $q_2 = q_3 = \cdots = q_n > q_1$ the first term on the right in (4.2) reduces to $(q_n - q_1)(nq_n)^{-1}x_1$; therefore $D_n - \Delta_n/(nq_n) = 0$ if, and only if, $x_1 = x_1^{q_1}x_2^{q_2} \cdots x_n^{q_n}$. Combining this with the above condition resulting from the lemma we deduce that $x_1 = 0$, $x_2 = x_3 = \cdots = x_n > 0$, which completes the proof.

5. On Hölder's inequality. We employ a well-known argument.[4] Set $a_{ik}/s_k = b_{ik}$; then $b_{1k} + b_{2k} + \cdots + b_{mk} = 1$ for each k. Now

$$1 - \sum_{i=1}^{m} b_{i1}^{q_1} b_{i2}^{q_2} \cdots b_{in}^{q_n} = \sum_{i=1}^{m} \left(q_1b_{i1} + q_2b_{i2} + \cdots - q_nb_{in} + \prod_{k=1}^{n} b_{ik}^{q_k} \right).$$

We can apply Theorem 1 to each of the expressions in brackets. Thus

[4] Loc. cit. §2.7, proof (ii) of Hölder's inequality.

$$
(5.1)\quad \begin{aligned} &\frac{q_1}{n-1}\sum_{i=1}^{m}\sum_{1\leqq j<k\leqq n}((b_{ij})^{1/2}-(b_{ik})^{1/2})^2 \\ &\leqq 1-\sum_{i=1}^{m}\prod_{k=1}^{n} b_{ik}^{q_k} \leqq \min\left\{1, q_n\sum_{i=1}^{m}\sum_{1\leqq j<k\leqq n}((b_{ij})^{1/2}-(b_{ik})^{1/2})^2\right\}, \end{aligned}
$$

which is equivalent to the result required. Clearly it implies that $\sum b_{i1}^{q_1}b_{i2}^{q_2}\cdots b_{in}^{q_n}=1$ if, and only if, for each i, j and k the numbers b_{ij} and b_{ik} are equal.

Henceforward we exclude this. In special cases certainly equality in (5.1) can be attained. For $n=2$, $m\geqq 2$, see §1. When $m=n$ and $q_1=q_2=\cdots=q_n=1/n$, for instance, there is equality on the left if and only if, for some permutation $\kappa, \lambda, \cdots, \omega$ of the numbers $1, 2, \cdots, n$, $b_{1\kappa}=b_{2\lambda}=\cdots=b_{m\omega}=1$; equality on the right if and only if $b_{1\kappa}=b_{2\lambda}=\cdots=b_{m\omega}=0$ while all the other b_{ik} are positive and equal (i.e., $=1/(m-1)$).

6. A general, yet preliminary, result on Minkowski's inequality.

THEOREM 3. *Let* $m\geqq 2$, $n\geqq 2$, $r>1$, $r'=r(r-1)^{-1}$, $a_{ik}\geqq 0$, $\rho_k^r=\sum_{i=1}^m a_{ik}^r>0$,

$$
S^r=\sum_{i=1}^{m}\left(\sum_{k=1}^{n} a_{ik}\right)^r,
$$

$$
L_k=\sum_{i=1}^{m}\left\{\left(\frac{a_{i1}+\cdots+a_{in}}{S}\right)^{r/2}-\left(\frac{a_{ik}}{\rho_k}\right)^{r/2}\right\}^2.
$$

Then

$$
(6.1)\qquad \frac{1}{\max(r, r')}\sum_{k=1}^{n}\rho_k L_k \leqq \sum_{k=1}^{n}\rho_k - S \leqq \frac{1}{\min(r, r')}\sum_{k=1}^{n}\rho_k L_k.
$$

Clearly there is equality whenever $r=2$; and $\sum_{k=1}^n \rho_k>S$, unless $a_{ik}/\rho_k=a_{il}/\rho_l$ for any i, k, l, i.e. unless the $a_{i1}, a_{i2}, \cdots$ are proportional.

The proof is based on a well-known argument.[5] If $\sigma_i=\sum_{k=1}^n a_{ik}$, $b_{ik}=a_{ik}^r$, then

$$
(6.2)\qquad S=S^{1-r}\left\{\sum_{i=1}^{m} a_{i1}\sigma_i^{r-1}+\sum_{i=1}^{m} a_{i2}\sigma_i^{r-1}+\cdots+\sum_{i=1}^{m} a_{in}\sigma_i^{r-1}\right\},
$$

$$
(6.3)\qquad a_{ik}\sigma_i^{r-1}=b_{ik}^{1/r}(\sigma_i^r)^{1/r'}.
$$

We fix k $(=1$, or $2, \cdots, n)$ and use Theorem 2. If $r>r'$ let $q_1=1/r$,

[5] Loc. cit., §2.11, first proof of Minkowski's inequality.

$q_2=1/r'$; $s_1=\sum_{i=1}^m b_{ik}=\rho_k^r$, $s_2=\sum_{i=1}^m \sigma_i^r=S^r$; we deduce that

$$(6.4)\qquad \rho_k S^{r-1}(1-q_2L_k) \leqq \sum_{i=1}^m b_{ik}^{1/r}(\sigma_i^r)^{1/r'} \leqq \rho_k S^{r-1}(1-q_1L_k).$$

If $r<r'$ interchange q_1 and q_2, and s_1 and s_2 also. Combining these results we can complete the proof. A theorem which is more suitable for applications will be given in another paper.

Appendix. Added on October 10, 1957. We prove the Remark (b) to Theorem 1; and in addition, two results: if $n=2$, $q_1\gtrless q_2$, the fraction in (1.2) tends to a unique limit when $(x_1-x_0)^2+(x_2-x_0)^2\to 0$, where $x_0>0$ is fixed; for $n\geqq 2$, there is never a unique limit when $x_0=0$, except for the trivial case $n=2$, $q_1=q_2=1/2$.

We set $U_n(x)=\sum((x_i)^{1/2}-(x_k)^{1/2})^2$ $(1\leqq i<k\leqq n)$ and take $\sum_i (x_i-x_0)^2\to 0$. We observe that not all the x_i are equal.

I. Let $n>2$, $q_1<q_n$. If $x_2=x_3=\cdots=x_n=0$, $x_1=x_0+\epsilon$ $(\epsilon\gtrless 0)$,

$$(1.1)\qquad \Delta_n(x)/(U_n(x)\to 2(n-1)^{-1}q_1(1-q_1) \qquad (\epsilon\to 0)$$

by the binomial series. If, however, $x_1=\cdots=x_{n-1}=0$, $x_n=x_0+\epsilon$, the limit is $2(n-1)^{-1}q_n(1-q_n)>2(n-1)^{-1}q_1(1-q_1)$. Thus there is no unique limit.

II. Let $n>2$, $q_1=q_2=\cdots=q_n=1/n$. We need only show the *existence* of a *unique* limit. Then (1.1) implies that the latter is $2n^{-2}$. Set $x_i=X_i^{2n}$ $(i=1,2,\cdots,n)$, $x_0=A^{2n}$. Then

$$(1.2)\quad U_n(X^{2n})=\sum_{1\leqq i<k\leqq n}(X_i-X_k)^2P_{ik};\qquad P_{ik}=\left(\sum_{j=0}^{n-1}X_i^{n-1-j}X_k^j\right)^2.$$

Hence, given a sufficiently small $\epsilon>0$, there is a δ such that

$$(1.3)\quad 0<n^2A^{2n-2}-\epsilon<P_{ik}<n^2A^{2n-2}+\epsilon,\quad \left\{\sum_i (X_i-A)^2<\delta\right\}.$$

Now it follows from the Hurwitz-Muirhead identity[6] that the polynomial $n\Delta_n(X^{2n})=\sum_i X_i^{2n}-nX_1^2X_2^2\cdots X_n^2$ is representable in the form

$$(1.4)\qquad n\Delta_n(X^{2n})=\sum_{1\leqq i<k\leqq n}(X_i-X_k)^2Q_{ik},$$

where the poynomials $Q_{ik}\equiv Q_{ki}$ are of degree $2n-2$ and are homogeneous in the variables $X_1, X_2, \cdots, X_n$, with all their coefficients positive. Let K be the sum of the coefficients of Q_{ik}. Since Q_{jl} may

[6] Loc. cit., §2.23 (2.23.1) and §2.21(1).

be derived from Q_{ik} by interchanging X_i with X_j and X_k with X_l, K is independent of i, k. Thus, for some $\delta>0$,

$$(1.5)\quad 0 < KA^{2n-2} - \epsilon < Q_{ik} < KA^{2n-2} + \epsilon, \qquad \left\{ \sum_i (X_i - A)^2 < \delta \right\}.$$

By (1.4) and (1.5),

$$\frac{KA^{2n-2} - \epsilon}{n^2 A^{2n-2} + \epsilon} < \frac{n\Delta_n}{U_n} < \frac{KA^{2n-2} + \epsilon}{n^2 A^{2n-2} - \epsilon}.$$

As $\epsilon \to 0$, $\Delta_n/U_n \to Kn^{-3}$; which shows the existence of the limit.

III. The case $n=2$. We show that the function

$$f(x, y; p) = (px + qy - x^p y^q)/(x^{1/2} - y^{1/2})^2 \qquad (0 < p < p + q = 1)$$

tends to $2pq$ as $(x-a)^2+(y-a)^2 \to 0$, where $a>0$ is fixed. If p is rational, this follows readily[7] from the preceding result.

Let now p be irrational and let $\{P_m\}$ and $\{p_m\}$ $(m=1, 2, \cdots)$ be sequences of rational numbers tending to p and such that $P_m > p > p_m$. Then $Q_m \to q$, $q_m \to q$ $(m \to \infty)$, where $Q_m = 1 - P_m$, $q_m = 1 - p_m$, and

$$(1.6)\quad \lim_{(x,y)\to(a,a)} f(x, y; P_m) = 2P_m Q_m; \qquad \lim_{(x,y)\to(a,a)} f(x, y; p_m) = 2p_m q_m.$$

(i) Let $x>y>0$, $x/y=u$ (>1), and consider the function $g(s) = sx + (1-s)y - x^s y^{1-s}$ for fixed x, y. Suppose first that $p<1/2$. We may take $P_m < 1/2$ $(m=1, 2, \cdots)$ and $s<1/2$. Then $2p_m q_m < 2pq < 2P_m Q_m$,

$$\begin{aligned} \frac{g'(s)}{y} &= u - 1 - u^s \log u > u - 1 - u^{1/2} \log u \\ &= u^{1/2} \int_1^u (v^{1/4} - v^{-1/4})^2 \frac{dv}{2v} \\ &> 0. \end{aligned}$$

Hence $g(P_m) > g(p) > g(p_n)$; as $g(p_m) = (x^{1/2} - y^{1/2})^2 f(p_m)$, by (1.6)

$$(1.7)\quad 2p_m q_m \leqq \liminf_{(x,y)\to(a,a)} f(x, y; p) \leqq \limsup_{(x,y)\to(a,a)} f(x, y; p) \leqq 2P_m Q_m.$$

Let now $p>1/2$. Then we may suppose that, for some constant $c>1/2$, $c<p_m<p<P_m<1$ $(m=1, 2, \cdots)$. Set $u_0 = \{c(1-c)^{-1}\}^{1/(1-c)}$. Then $u_0>1$,

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[7] Loc. cit., cf. §2.2 and the proof of (2.5.1).

$$\frac{g'(s)}{yu^c} \leqq \frac{u - 1 - u^c \log u}{u^c} = \int_1^u \left\{(1 - c)v + c - v^c\right\} \frac{dv}{v^{1+c}}$$

$$= c\int_1^u \frac{dv}{v^{1+c}}\int_1^v \left(\frac{1}{u_0^{1-c}} - \frac{1}{z^{1-c}}\right) dz;$$

which is certainly negative for $1 < u < u_0$. Since $x \to a$ and $y \to a$, we may take u, i.e. x/y, $< u_0$. Hence $f(x, y; P_m) < f(x, y; p) < f(x, y; p_m)$, and we obtain an inequality similar to (1.7), with $2p_m q_m$ and $2P_m Q_m$ interchanged. Taking $m \to \infty$, we have

$$\lim_{(x,y)\to(a,a)} f(x, y; p) = 2pq. \tag{1.8}$$

(ii) Let now $y > x$. We arrive at the same result when observing that $f(x, y; p) = f(y, x; q) \to 2qp$; thus we can complete the proof.

IV. The last statement is deduced by an argument similar to that used in I. Thus all the assertions are true.

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