ON TRANSFORMATIONS PRESERVING LAGUERRE-FORSYTH CANONICAL FORM[1]

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1. **Introduction.** Consider the linear differential equation

$$(1.1)\quad y^{(n)} + C_{n,1}P_1(s)y^{(n-1)} + C_{n,2}P_2(s)y^{(n-2)} + \cdots + C_{n,n}P_n(s)y = 0,$$

where the $P_i(s)$ are regular functions of s in $-\infty < s < \infty$, the $C_{n,i}$ are binomial coefficients, and the derivatives $y^{(i)}$ are taken with respect to s. The n linearly independent solutions $y_1(s), \cdots, y_n(s)$ of (1.1), considered as homogeneous coordinates of a point $y(s)$, define an integral curve C parametrized by s and immersed in an $(n-1)$-dimensional projective space S_{n-1}. C is unique up to a projective transformation. Lane [2] and Wilczynski [3], among others, have used analytic properties of (1.1) to investigate projective properties of C, frequently reducing (1.1) by appropriate transformations of y and s to Laguerre-Forsyth (LF) canonical form so as to simplify their results. This form is characterized by the absence of the derivatives of order $n-1$ and $n-2$:

$$(1.2)\quad x^{(n)} + C_{n,3}p_3(t)x^{(n-3)} + C_{n,4}p_4(t)x^{(n-4)} + \cdots + C_{n,n}p_n(t)x = 0,$$

where the $p_i(t)$ are regular functions of t in $-\infty < t < \infty$. Since there is an infinite number of ways in which reduction to canonical form may be accomplished there is no unique LF canonical form for (1.1). However it may be shown that both reduction of (1.1) to a LF form and transformation from one LF form to another leave the projective properties of C invariant, the change being essentially in its parametrization. Also invariant under such operations are the (linear) osculating spaces $S_k(s)$ $(0 \leqq k \leqq n-1)$ at each point $y(s)$ of C. As is well-known [2, Chapter 1], each $S_k(s)$ has a basis consisting of the first $k+1$ derivative points $y/0$ $(=y)$, $y/1, \cdots, y/k$, the / indicating differentiation of each coordinate function of $y(s)$ with respect to s, and the number the order of the derivative. These points however are in general functions of the parametrization and so do not remain invariant under reduction to canonical form or even under the continuous group of transformations preserving canonical form. Their loci in $S_{n-1}(=S_{n-1}(s)$ for all $s)$ as the transformation preserving

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canonical form is varied continuously are calculated in §2 and given an intrinsic interpretation in §3.

2. **The loci C_k.** Without loss of generality we begin with (1.2). Let $x/(n-1)$ $(=d^{n-1}x/dt^{n-1})$, $x/(n-2), \cdots, x/1, x/0$ be the independent derivative points associated with this canonical form. LF canonical form is preserved under the elements of the continuous group T of transformations of the following type:

$$\bar{t} = (\alpha t + \beta)/(\gamma t + \delta); \qquad \bar{x} = \lambda(d\bar{t}/dt)^{(n-1)/2}x, \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta, \lambda$ are real constants and $\alpha\delta-\beta\gamma\neq 0$, [**3**, p. 26]. The structure of these transformations shows that if (1.2) is subjected to any element of T, the derivative point $\bar{x}/k$ (the derivative now being with respect to $\bar{t}$) associated with the new canonical form is a linear combination of $x/0, \cdots, x/k$ in which the coefficient of x/k is not zero. Hence $\bar{x}/k$ is in $S_k(t)$ but not in $S_j(t)$ for $j<k$.

To study the locus C_k of $\bar{x}/k$ in $S_k(t)$ when t is fixed and (2.1) varies continuously first suppose that $\gamma\neq 0$ and consider the subset of T whose elements have the form

$$\bar{t} = \mu/(t + \omega) + \nu; \qquad \bar{x} = x\rho/(t+\omega)^{n-1} \tag{2.2}$$

where $\omega=\delta/\gamma$, $\nu=\alpha/\gamma$, $\mu=-(\alpha\delta-\beta\gamma)/\gamma^2$, $\rho=\lambda(-\mu)^{(n-1)/2}$. From (2.2) $\bar{x}/k$ is calculated in terms of $x/0, \cdots, x/k$ by differentiating the expression for $\bar{x}$ with respect to t and using the chain rule for derivatives with $dt/d\bar{t}$ calculated from the expression for $\bar{t}$. An inductive argument shows that the coefficient of $x/(k-i)$ in the expression for $\bar{x}/k$ is given by

$$\begin{aligned}\overset{k}{\nu}_{k-i} = (\rho/\mu^{k})(-1)^{k-i}C_{k,k-i}(n-k)(n-k+1)\cdots \\ \cdots(n-k+(i-1))\tau^{n-2k+(i-1)}\end{aligned} \tag{2.3}$$

$(i=0, 1, \cdots, k;\ k=1, 2, \cdots, n-1)$, where $\tau=1/(t+\omega)$. Since the points $x/0, \cdots, x/k$ form a basis for $S_k(t)$, the coefficients (2.3) can be taken as homogeneous local coordinates of $\bar{x}/k$ in $S_k(t)$, and the factor (ρ/μ^k), being common, may then be divided out. Now the coefficients are functions of τ and so of ω since t is fixed. When τ is eliminated between the $k+1$ coefficients, equations in the local coordinates arise which represent surfaces in $S_k(t)$. Since there are $k(k-1)/2$ ways of eliminating τ we get a set of surfaces defining C_k by their intersections. Thus, from the ratios

$$\frac{\overset{k}{\nu}_{k-(i+1)}}{\overset{k}{\nu}_{k-i}} = (-1)\frac{k-i}{i+1}\,\frac{n-k+i}{1}\,\tau \qquad (i = 0, 1, \cdots, k-1)$$

we get, on division of consecutive ones, the surfaces

$$(2.4)\qquad \frac{\overset{k}{\nu}_{k-(i+1)}\overset{k}{\nu}_{k-(i-1)}}{(\nu^k_{k-i})^2} = \frac{i}{i+1}\,\frac{k-i}{k-i+1}\,\frac{n-k+i}{n-k+(i-1)},\qquad (i = 1, 2, \cdots, k-1).$$

The remaining surfaces are had in a similar manner.

Finally, if $\gamma=0$, we may suppose that $\delta=1$. Then $\bar{t}=\alpha t+\beta$ and $\bar{x}=\lambda\alpha^{(n-1)/2}x$. As a result $\bar{x}/k=\lambda\alpha^{(n-2k-1)/2}x/k$ $(k=0, 1, \cdots, n)$ so that in this case the derivative points are invariant under continuous variation of the transformation.

3. **Intrinsic interpretation of C_k.** The locus C_k admits an intrinsic interpretation. Specifically, the locus C_k^* traced by the point of intersection of $S_{n-1-k}(t+\epsilon)$ of a nearby point $x(t+\epsilon)$ of C and $S_k(t)$ at $x(t)$ coincides with C_k in the limit as $\epsilon\rightarrow 0$. The proof is as follows. First, $x(t+\epsilon)$ is expanded around t in a Taylor's series in ϵ whose coefficients are multiples of the derivatives of $x(t)$. This is always possible since $x(t)$ is the solution of an equation with regular coefficients. Through the canonical form all derivatives of order n or greater are replaced by linear combinations of derivatives of order $n-1$ and less. When the terms are grouped according to the order of the derivative appearing we get a representation of $x(t+\epsilon)$ as a point in S_{n-1} in terms of $x/0, \cdots, x/(n-1)$, where the coefficient of x/s is $\nu_s=(\epsilon^s/s!)-(\epsilon^n/n!)\Sigma_s$ where Σ_s is a power series in ϵ whose form will not enter the argument. Put $y(\epsilon)\equiv x(t+\epsilon)$. Then any point $\bar{y}$ in $S_{n-1-k}(t+\epsilon)$ can be written

$$(3.1)\qquad \bar{y} = \sum_{\alpha=0}^{n-1-k} (C_\alpha\epsilon^\alpha)y/\alpha,\qquad (y/\alpha \equiv d^\alpha y/d\epsilon^\alpha),$$

where the C_α's are variable. When the expression for $y(\epsilon)$ is introduced into (3.1) we get

$$\bar{y} = \left(C_0\nu_0 + \sum_{\alpha=1}^{n-1-k} C_\alpha\epsilon^\alpha\nu_0/\alpha\right)x + \sum_{s=1}^{n-1}\left(C_0\nu_s + \sum_{\alpha=1}^{n-1-k} C_\alpha\epsilon^\alpha\nu_s/\alpha\right)x/s,$$
$$(\nu_s/\alpha \equiv d^\alpha\nu_s/d\epsilon^\alpha),$$

which expresses $\bar{y}$ as a point in S_{n-1}. Hence a necessary and sufficient condition that $\bar{y}$ be the point of intersection of $S_{n-1-k}(t+\epsilon)$ and $S_k(t)$ is

$$(3.2)\qquad C_0\nu_s + \sum_{\alpha=1}^{n-1-k} C_\alpha\epsilon^\alpha\nu_s/\alpha = 0,\qquad (s = k+1, \cdots, n-1).$$

System (3.2) has a nontrivial solution in the C_α's involving an arbitrary nonzero constant C_0. If ϵ is limited to small absolute values so that $\nu_s \doteq \epsilon^s/s!$, then (3.2) is equivalent to

$$\sum_{\alpha=0}^{n-1-s} \frac{1}{(n-s-1-\alpha)!} C_\alpha = 0, \quad (s = 0, 1, \cdots, n-1-(k+1)),$$

whose solutions are given by

$$C_\alpha = (-1)^\alpha \frac{(n-1-\alpha)(n-2-\alpha)\cdots(n-k-\alpha)}{\alpha!(n-1)(n-2)\cdots(n-k)} C_0,$$
$$(\alpha = 0, 1, \cdots, n-1),$$

where C_0 is arbitrary. On the substitution of these values into

$$\bar{y} = \left(C_0\nu_0 + \sum_{\alpha=1}^{n-1-k} C_\alpha\epsilon^\alpha\nu_0/\alpha\right)x + \sum_{s=1}^{k}\left(C_0\nu_s + \sum_{\alpha=1}^{n-1-k} C_\alpha\epsilon^\alpha\nu_s/\alpha\right)x/s,$$

we find that the coefficients of $x/(k-i)$ are given by

$$\overset{k}{\mu}_{k-i} = (\epsilon^{k-i}/(k-i)!) \sum_{\alpha=0}^{k-i} (-1)^\alpha C_{k-i,\alpha} \frac{(n-1-\alpha)\cdots(n-k-\alpha)}{(n-1)\cdots(n-k)},$$

so that, by a result due to Cauchy (see [1, Formula 13, p. 67]),

$$\overset{k}{\mu}_{k-i} = \epsilon^{k-i} \frac{C_{k,k-i}}{(n-1)(n-2)\cdots(n-k+i)}.$$

These μ^k_{k-i} are interpreted as local homogeneous coordinates in $S_k(t)$ of the point of intersection, and are continuous functions of the parameter ϵ. Elimination of the parameter ϵ now gives precisely the surfaces of §2. This completes the proof.

References

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