ON TRANSITIVE TRANSLATION FUNCTIONS

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From the definition\(^2\) of a fiber space \((E, B, p)\) in terms of a lifting function,

\[
\lambda: \{(e, \omega) \in E \times B' \mid p(e) = \omega(0)\} \rightarrow E' \text{ such that } p \circ \lambda(e, \omega) = \omega,
\]

we are led to a translation function

\[
\tau: \{(e, \omega) \mid p(e) = \omega(0)\} \rightarrow E \text{ where } \tau(e, \omega) = \lambda(e, \omega)(1).
\]

We may also consider the maps \(\tau(\omega): p^{-1}(\omega(0)) \rightarrow p^{-1}(\omega(1))\) defined by \(\tau(\omega)(e) = \tau(e, \omega)\). A translation function is transitive if \(\tau(\omega_1 \cdot \omega_2) = \tau(\omega_2) \circ \tau(\omega_1)\) where

\[
\omega_1(1) = \omega_2(0) \text{ and } (\omega_1 \cdot \omega_2)(t) = \begin{cases} 
\omega_1(2t) & \text{for } 0 \leq t \leq 1/2, \\
\omega_2(2t - 1) & \text{for } 1/2 \leq t \leq 1.
\end{cases}
\]

The question of when transitive translation functions exist for fiber bundles was raised by W. Hurewicz. The answer this paper supplies is the following.

If a bundle over a finite polyhedron has a structural group \(G\) with no small subgroups then it has a transitive translation function if and only if it is equivalent in \(G\) to an \(H\) bundle where \(H\) is a totally disconnected subgroup of \(G\).

The central result of this paper is that if \(\tau\) is a transitive translation function and the structural group has no small subgroups, then \(\tau(\omega)\) depends only on the homotopy class of \(\omega\).

All spaces we consider will be Hausdorff spaces; path spaces will have the compact-open topology.

Remarks. For path spaces one may take as a basis all sets of the form \(N = \bigcap_{i=1}^{2^n} \left(\left[\frac{(i-1)/2^n}{i/2^n}\right], U_i\right)\).

A sequence \(\omega_n\) converges to the constant \(x_0\) in the path space \(X'\) if and only if every neighborhood \(U\) of \(x_0\) in \(X\) contains all but a finite number of the sets \(\omega_n(I)\). That this is not true for Moore paths pre-

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vents the extension of our results to translation functions defined on Moore paths.

**Definitions.** A triple \((E, B, p)\) is a regular fiber space if there exists a function \(\tau: \{(e, \omega) \mid p(e) = \omega(0)\} \to E\) such that \(p(\tau(e, \omega)) = \omega(1)\) and \(\tau(e, \omega) = e\) if \(\omega(1) = e\). A fiber space is homeomorphic if each of the maps \(\tau(\omega)\) is a homeomorphism.

A fiber space is transitive if \(\tau(\omega_1 \cdot \omega_2) = \tau(\omega_2) \circ \tau(\omega_1)\).

In the case of a fiber bundle \((E, B, p, Y, G)\) over a paracompact base space \(B\), the Hurewicz uniformization theorem gives the result: \((E, B, p)\) is a regular homeomorphic fiber space and \(\tau(\omega)\) may be identified with a member of \(G\) through any two applicable coordinate maps. We shall therefore define a translation function \(\tau\) for the bundle \((E, B, p, Y, G)\) as a regular homeomorphic translation function for \((E, B, p)\) with the property that \(\phi_i^{-1}(\omega(1))\tau(\omega)\phi_j(\omega(0)) \in G\) for some pair (and therefore all pairs) of coordinate functions \(\phi_i, \phi_j\) such that \((\omega(1), \omega(0)) \in V_i \times V_j\).

**Definition.** \(\omega^*\) is a reparametrization of \(\omega\) if there exists a sense preserving homeomorphism \(f\) of the unit interval onto itself such that \(\omega^* = \omega \circ f\).

**Lemma.** If \(\omega^*\) is a reparametrization of \(\omega\) and \(\tau\) a transitive translation function, then \(\tau(\omega) = \tau(\omega^*)\).

From the associativity of the composition of maps we know that \(\tau(\omega_1 \cdots \omega_n) = \tau(\omega_n) \circ \cdots \circ \tau(\omega_1)\) is independent of the bracketting of \(\omega_1 \cdots \omega_n\). In this proof we shall approximate \(\omega^*\) by a path \(\omega'\) which is obtained from \(\omega\) by letting \(\omega = \omega_1 \cdots \omega_2^m\) in the canonical bracketting (defined below) and \(\omega' = \omega_1 \cdots \omega_2^m\) in another bracketting. The continuity of \(\tau\) then gives the result. We shall use the phrase “\(\omega = \omega_1 \cdots \omega_2^n\) where the bracketting is canonical” to mean \(\omega = \tilde{\omega} \cdot \bar{\omega}\) where \(\tilde{\omega} = \omega_1 \cdots \omega_2^{m-1}\) and \(\bar{\omega} = \omega_2^{m-1+1} \cdots \omega_2^n\) where in each case the bracketting is canonical. The canonical bracketting is defined only when the number of factors is a power of 2.

**Proof of Lemma.** Let \(\omega^*(t) = \omega(f(t))\) and \(N\) be a neighborhood of \(\omega^*\). We may assume \(N = \cap_{i=1}^{2^n} \left([((i-1)/2^n, i/2^n], U_i\right)\). Let \(U_i\) be a connected neighborhood of \([((i-1)/2^n, i/2^n], C_i\) such that \(\omega^*(U_i) \subseteq U_i\). Choose dyadic rationals \(0 < m_1/2^m < \cdots < m_2^n/2^m = 1\) such that \(m_i/2^m \in \cap f(U_i) \cap f(U_{i+1})\). We shall now construct a path \(\omega'\), which will send \(C_i\) into \(\omega([m_{i-1}/2^m, m_i/2^m]) \subseteq \omega^*(U_i) \subseteq U_i\).

Let \(\omega = \omega_1 \cdots \omega_2^m\) where the bracketting is canonical. Let \(\omega'_i = \omega_m \cdots \omega_2^m\) where the bracketting is arbitrary. Define \(\omega' = \omega'_1 \cdots \omega'_2^m\) where the bracketting is canonical. Now \(\omega'(C_i) = \omega'_i(I)\).
\[= \omega_{m_{i-1}+1} \cdot \cdots \cdot \omega_{m_i}(I) = \omega(\{m_{i-1}/2^m, m_i/2^m\}) \subseteq U_i; \text{ therefore } \omega' \in N \]

which proves the result. q.e.d.

In the remainder of this paper we shall assume that the structural group \( G \) has the following property:

If \( \omega \in \mathcal{G}', \omega(0) = e \) the identity, then either

1. There exists a sequence \( t_i \) of reals from the unit interval converging to zero and positive integers \( m_i \) such that \( (\omega(t_i))^{m_i} \) does not converge to the identity, or

2. There exists a \( t_0 \) such that \( \omega(t) = e \) for \( t \leq t_0 \).

In particular groups with no small subgroups have this property.

**Theorem.** If \( \tau \) is a transitive lifting function for the bundle \( (E, B, p, Y, G) \), then the map \( \tau(\omega) \) depends only on the homotopy class of \( \omega \).

We proceed by a series of lemmas.

**Lemma 1.** If \( \omega_s(t) \) is a homotopy of the loop \( \omega_1(t) \) at \( b_0 \) to the constant loop \( \omega_0(t) = b_0 \), then there exists a number \( s_0 > 0 \) such that \( \tau(\omega_s) = e \), the identity, for \( s \leq s_0 \).

**Proof.** \( \tau(\omega_s) \) is a path in \( G \) such that \( \tau(\omega_0) = e \). Since condition (2) is our lemma, it suffices to show that condition (1) is impossible. Let \( (t_i, m_i) \) be a sequence of reals and positive integers, as in condition (1). By the transitivity of \( \tau \) we have \( (\tau(\omega_{t_i}))^{m_i} = \tau(\omega_{t_i}^{m_i}) \) where \( \omega_{t_i}^{m_i} \) is defined by \( \omega_{t_i+1} = \omega_t \cdot \omega_t \). Since \( t_i \) converges to zero, we have that \( \omega_{t_i}^{m_i} \) converges to the constant path \( \omega_0 \). Since \( \tau \) is continuous, \( \tau(\omega_{t_i}^{m_i}) \) converges to the identity; thus (1) cannot hold. q.e.d.

If \( \omega(t) \) is a path, let \( \omega(t)^{-1} \) denote the path defined by \( \omega(t)^{-1}(t) = \omega(1-t) \).

**Lemma 2.** \( (\tau(\omega))^{-1} = (\tau(\omega^{-1})) \).

**Proof.** Let \( \omega_s(t) = \omega(st) \). Then \( \omega_s \cdot \omega_s^{-1} \) is a homotopy of a loop to the constant. Let \( s_0 = \sup \{ s \in I | \tau(\omega'_t, \omega_t^{-1}) = e \text{ for } t \leq s \} \). By Lemma 1 \( s_0 > 0 \); we shall show by a contradiction that \( s_0 = 1 \).

If \( s_0 < 1 \), then \( \omega_s \cdot \omega' \) is a reparametrization of \( \omega \) where \( \omega'(t) = \omega(s_0 + (1-s_0)t) \). Consider \( \omega'_s(t) = \omega'(st) \) and \( \tau(\omega'_s, \omega'_s^{-1}) \). Lemma 1 gives us \( s_1 > 0 \) such that \( \tau(\omega'_s \cdot \omega'_s^{-1}) = e \) for \( s < s_1 \). However \( \omega_{s_0} \cdot \omega'_s \) is a reparametrization of \( \omega_{s_0-r-r} \). Thus

\[
\tau(\omega_{s_0+r-r}, \omega_{s_0+r-r})^{-1} = \tau((\omega_{s_0} \cdot \omega'_s) \cdot (\omega_{s_0} \cdot \omega'_s)^{-1})
= \tau(\omega_{s_0}^{-1} \circ \tau(\omega'_s^{-1}) \circ \tau(\omega'_s) \circ \tau(\omega_{s_0}))
= e \text{ the identity for } r \leq s_1.

Hence \( \tau(\omega_s, \omega_s^{-1}) = e \text{ for } s \leq s_0 + s_1 - s_0 s_1 \). Since \( s_0 \) is maximal we have
$s_0 + s_1 - s_0s_1 \leq s_0$ or $1 \leq s_0$ which contradicts $s_0 < 1$. q.e.d.

We are now in a position to prove the theorem. The technique is similar to the proof of Cauchy's theorem. Let $\Delta$ denote the model two-simplex and $\Delta_i^n$ the $i$th simplex of the $n$th barycentric subdivision. Choose loops $\rho_{n,i}$ which are (clockwise) homeomorphisms of the reals modulo 1 onto the boundary of $\Delta_i^n$. By the convexity of $\Delta$ define the path $\rho_{x,n,i}(t) = (1-t)x + t\rho_{n,i}(0)$ where $x \in \Delta$. Let $\partial_\pm \Delta_i^n$ denote the loop $\rho_{x,n,i}(\rho_{n,i}^{-1})$.

**Proof of Theorem by Contradiction.** Let $\omega$ be a null homotopic loop such that $\tau(\omega) \neq e$ the identity. Let $\sigma: \Delta \to B$ be a singular simplex such that $\sigma \circ \partial_0 \Delta = \omega$. There must be a simplex $\Delta_i^0$ of the first barycentric subdivision such that $\tau(\sigma \circ \partial_0 \Delta_i^0) \neq e$ for otherwise $\tau(\omega) = e$. We continue by induction and find a nested sequence of simplexes $\Delta_i^n$ such that $\tau(\sigma \circ \partial_0 \Delta_i^n) \neq e$. Let $x = \bigcap_{n=0}^{\infty} \Delta_i^n$. Let $\rho_x(t)$ be a homotopy of loops at $x$ such that $\rho_{x,n} = \partial_\pm \Delta_i^n$. Then by Lemma 1 there exists $s_0 > 0$ such that $\tau(\sigma \circ \rho_x) = e$ for $s \leq s_0$; in particular there is some $n$ such that $\tau(\sigma \circ \partial_\pm \Delta_i^n) = e$. But this implies $\tau(\sigma \circ \partial_\pm \Delta_i^n) = e$ which is the desired contradiction. q.e.d.

**Theorem.** A bundle over a finite polyhedron with a structural group $G$, which has no small subgroups, has a transitive translation function if and only if it is equivalent in $G$ to an $H$ bundle where $H$ is a totally disconnected subgroup of $G$.

**Proof.** Let $\tau$ be the translation function that exists by the Hurewicz uniformization theorem for the $H$ bundle. Then since the maps $\tau(\omega_1 \cdot \omega_2)$ and $\tau(\omega_2) \circ \tau(\omega_1)$ are homotopic and $H$ is totally disconnected it follows that they are equal. Hence $\tau$ is transitive.

If, on the other hand, there is a transitive translation function $\tau$, we can construct a coordinate bundle with a totally disconnected group. Select a point $b_0$ in the base space and let $\phi_0(b_0): Y \to p^{-1}(b_0)$ be one of the coordinate maps restricted to $Y \times b_0$. For every vertex $a_i$ select a contraction of its star to the point $b_0$. Thus for every $x \in \operatorname{St} a_i$ we obtain a path $\omega_{a_i,x}$ from $b_0$ to $x$. Define the maps $\phi_{a_i}: Y \times \operatorname{St} a_i \to p^{-1}(\operatorname{St} a_i)$ by $\phi_{a_i}(y, x) = \tau(\phi_0(y), \omega_{a_i,x})$. Thus we obtain coordinate maps which are compatible with the original coordinate maps. The subgroup $H$ of $G$ which is spanned by the $g_{a_i,a_j}(x)$, $x \in \operatorname{St} a_i \cap \operatorname{St} a_j$, is the continuous image of the finitely generated fundamental group of the base space and hence $H$ is totally disconnected. q.e.d.