ON TRANSITIVE TRANSLATION FUNCTIONS[1]

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From the definition[2] of a fiber space (E, B, p) in terms of a lifting function,

$$\lambda: \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\} \to E^I \text{ such that } p \circ \lambda(e, \omega) = \omega,$$

we are led to a translation function

$$\tau: \{(e, \omega) \mid p(e) = \omega(0)\} \to E \quad \text{where} \quad \tau(e, \omega) = \lambda(e, \omega)(1).$$

We may also consider the maps $\tau(\omega): p^{-1}(\omega(0)) \to p^{-1}(\omega(1))$ defined by $\tau(\omega)(e) = \tau(e, \omega)$. A translation function is transitive if $\tau(\omega_1 \cdot \omega_2) = \tau(\omega_2) \circ \tau(\omega_1)$ where

$$\omega_1(1) = \omega_2(0) \quad \text{and} \quad (\omega_1 \cdot \omega_2)(t) = \begin{cases} \omega_1(2t) & \text{for } 0 \leqq t \leqq 1/2, \\ \omega_2(2t - 1) & \text{for } 1/2 \leqq t \leqq 1. \end{cases}$$

The question of when transitive translation functions exist for fiber bundles was raised by W. Hurewicz. The answer this paper supplies is the following.

If a bundle over a finite polyhedron has a structural group G with no small subgroups then it has a transitive translation function if and only if it is equivalent in G to an H bundle where H is a totally disconnected subgroup of G.

The central result of this paper is that if τ is a transitive translation function and the structural group has no small subgroups, then $\tau(\omega)$ depends only on the homotopy class of ω.

All spaces we consider will be Hausdorff spaces; path spaces will have the compact-open topology.

REMARKS. For path spaces one may take as a basis all sets of the form $N = \bigcap_{i=1}^{2^n} ([(i-1)/2^n, i/2^n], U_i)$.

A sequence ω_n converges to the constant x_0 in the path space X^I if and only if every neighborhood U of x_0 in X contains all but a finite number of the sets $\omega_n(I)$. That this is not true for Moore paths pre-

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[2] W. Hurewicz, *On the concept of fiber space*, Proc. Nat. Acad. Sci. U.S.A. vol. 41, no. 11, pp. 956–961.

vents the extension of our results to translation functions defined on Moore paths.

DEFINITIONS. A triple (E, B, p) is a regular fiber space if there exists a function $\tau: \{(e, \omega) | p(e) = \omega(0)\} \to E$ such that $p(\tau(e, \omega)) = \omega(1)$ and $\tau(e, \omega) = e$ if $\omega(t) \equiv e$. A fiber space is homeomorphic if each of the maps $\tau(\omega)$ is a homeomorphism.

A fiber space is transitive if $\tau(\omega_1 \cdot \omega_2) = \tau(\omega_2) \circ \tau(\omega_1)$.

In the case of a fiber bundle (E, B, p, Y, G) over a paracompact base space B, the Hurewicz uniformization theorem gives the result: (E, B, p) is a regular homeomorphic fiber space and $\tau(\omega)$ may be identified with a member of G through any two applicable coordinate maps. We shall therefore define a translation function τ for the bundle (E, B, p, Y, G) as a regular homeomorphic translation function for (E, B, p) with the property that $\phi_i^{-1}(\omega(1))\tau(\omega)\phi_j(\omega(0)) \in G$ for some pair (and therefore all pairs) of coordinate functions ϕ_i, ϕ_j such that $(\omega(1), \omega(0)) \in V_i \times V_j$.

DEFINITION. ω^* is a reparametrization of ω if there exists a sense preserving homeomorphism f of the unit interval onto itself such that $\omega^* = \omega \circ f$.

LEMMA. *If ω^* is a reparametrization of ω and τ a transitive translation function, then $\tau(\omega) = \tau(\omega^*)$.*

From the associativity of the composition of maps we know that $\tau(\omega_1 \cdots \omega_n) = \tau(\omega_n) \circ \cdots \circ \tau(\omega_1)$ is independent of the bracketting of $\omega_1 \cdots \omega_n$. In this proof we shall approximate ω^* by a path ω' which is obtained from ω by letting $\omega = \omega_1 \cdots \omega_{2^m}$ in the canonical bracketting (defined below) and $\omega' = \omega_1 \cdots \omega_{2^m}$ in another bracketting. The continuity of τ then gives the result. We shall use the phrase "$\omega = \omega_1 \cdots \omega_{2^n}$ where the bracketting is canonical" to mean $\omega = \bar{\omega} \cdot \tilde{\omega}$ where $\bar{\omega} = \omega_1 \cdots \omega_{2^{n-1}}$ and $\tilde{\omega} = \omega_{2^{n-1}+1} \cdots \omega_{2^n}$ where in each case the bracketting is canonical. The canonical bracketting is defined only when the number of factors is a power of 2.

PROOF OF LEMMA. Let $\omega^*(t) = \omega(f(t))$ and N be a neighborhood of ω^*. We may assume $N = \bigcap_{i=1}^{2^n} ([(i-1)/2^n, i/2^n], U_i)$. Let $\mathcal{V}_i$ be a connected neighborhood of $[(i-1)/2^n, i/2^n] = C_i$ such that $\omega^*(\mathcal{V}_i) \subseteq U_i$. Choose dyadic rationals $0 < m_1/2^m < \cdots < m_{2^n}/2^m = 1$ such that $m_i/2^m \in f(\mathcal{V}_i) \cap f(\mathcal{V}_{i+1})$. We shall now construct a path ω', which will send C_i into $\omega([m_{i-1}/2^m, m_i/2^m]) \subseteq \omega^*(\mathcal{V}_i) \subseteq U_i$.

Let $\omega = \omega_1 \cdots \omega_{2^m}$ where the bracketting is canonical. Let $\omega_i' = \omega_{m_{i-1}+1} \cdots \omega_{m_i}$ where the bracketting is arbitrary. Define $\omega' = \omega_1' \cdots \omega_{2^n}'$ where the bracketting is canonical. Now $\omega'(C_i) = \omega_i'(I)$

$=\omega_{m_{i-1}+1}\cdots\omega_{m_i}(I)=\omega([m_{i-1}/2^m,\ m_i/2^m])\subseteq U_i$; therefore $\omega'\in N$ which proves the result. q.e.d.

In the remainder of this paper we shall assume that the structural group G has the following property:

If $\omega\in G^I$, $\omega(0)=e$ the identity, then either

(1) There exists a sequence t_i of reals from the unit interval converging to zero and positive integers m_i such that $(\omega(t_i))^{m_i}$ does not converge to the identity, or

(2) There exists a $t_0>0$ such that $\omega(t)=e$ for $t\leqq t_0$.

In particular groups with no small subgroups have this property.

THEOREM. *If τ is a transitive lifting function for the bundle (E, B, p, Y, G), then the map $\tau(\omega)$ depends only on the homotopy class of ω.*

We proceed by a series of lemmas.

LEMMA 1. *If $\omega_s(t)$ is a homotopy of the loop $\omega_1(t)$ at b_0 to the constant loop $\omega_0(t)=b_0$, then there exists a number $s_0>0$ such that $\tau(\omega_s)=e$, the identity, for $s\leqq s_0$.*

PROOF. $\tau(\omega_s)$ is a path in G such that $\tau(\omega_0)=e$. Since condition (2) is our lemma, it suffices to show that condition (1) is impossible. Let (t_i, m_i) be a sequence of reals and positive integers, as in condition (1). By the transitivity of τ we have $(\tau(\omega_{t_i}))^{m_i}=\tau(\omega_{t_i}^{m_i})$ where $\omega_{t_i}^{m_i}$ is defined by $\omega^{n+1}=\omega^n\cdot\omega$. Since t_i converges to zero, we have that $\omega_{t_i}^{m_i}$ converges to the constant path ω_0. Since τ is continuous, $\tau(\omega_{t_i}^{m_i})$ converges to the identity; thus (1) cannot hold. q.e.d.

If $\omega(t)$ is a path, let $\omega^{-1}(t)$ denote the path defined by $\omega^{-1}(t)=\omega(1-t)$.

LEMMA 2. $(\tau(\omega))^{-1}=(\tau(\omega^{-1}))$.

PROOF. Let $\omega_s(t)=\omega(st)$. Then $\omega_s\cdot\omega_s^{-1}$ is a homotopy of a loop to the constant. Let $s_0=\sup\{s\in I \mid \tau(\omega_t\cdot\omega_t^{-1})=e \text{ for } t\leqq s\}$. By Lemma 1 $s_0>0$; we shall show by a contradiction that $s_0=1$.

If $s_0<1$, then $\omega_{s_0}\cdot\omega'$ is a reparametrization of ω where $\omega'(t)=\omega(s_0+(1-s_0)t)$. Consider $\omega_s'(t)=\omega'(st)$ and $\tau(\omega_s'\cdot\omega_s'^{-1})$. Lemma 1 gives us $s_1>0$ such that $\tau(\omega_s'\cdot\omega_s'^{-1})=e$ for $s<s_1$. However $\omega_{s_0}\cdot\omega_r'$ is a reparametrization of $\omega_{s_0-r-rs_0}$. Thus

$$\begin{aligned}\tau(\omega_{s_0+r-rs_0}\cdot\omega^{-1}_{s_0+r-rs_0}) &= \tau((\omega_{s_0}\cdot\omega_r')\cdot(\omega_{s_0}\cdot\omega_r')^{-1})\\ &= \tau(\omega_{s_0}^{-1})\circ\tau(\omega_r'^{-1})\circ\tau(\omega_r')\circ\tau(\omega_{s_0})\\ &= e \text{ the identity for } r\leqq s_1.\end{aligned}$$

Hence $\tau(\omega_s\cdot\omega_s^{-1})=e$ for $s\leqq s_0+s_1-s_0s_1$. Since s_0 is maximal we have

$s_0+s_1-s_0s_1\leqq s_0$ or $1\leqq s_0$ which contradicts $s_0<1$. q.e.d.

We are now in a position to prove the theorem. The technique is similar to the proof of Cauchy's theorem. Let Δ denote the model two-simplex and Δ_i^n the ith simplex of the nth barycentric subdivision. Choose loops $\rho_{n,i}$ which are (clockwise) homeomorphisms of the reals modulo 1 onto the boundary of Δ_i^n. By the convexity of Δ define the path $\rho_{x,n,i}(t)=(1-t)x+t\rho_{n,i}(0)$ where $x\in\Delta$. Let $\partial_x\Delta_i^n$ denote the loop $\rho_{x,n,i}\cdot(\rho_{n,i}\cdot\rho_{x,n,i}^{-1})$.

Proof of Theorem by contradiction. Let ω be a null homotopic loop such that $\tau(\omega)\neq e$ the identity. Let $\sigma:\Delta\rightarrow B$ be a singular simplex such that $\sigma\circ\partial_0\Delta=\omega$. There must be a simplex Δ_{i_1}' of the first barycentric subdivision such that $\tau(\sigma\circ\partial_0\Delta_{i_1}')\neq e$ for otherwise $\tau(\omega)=e$. We continue by induction and find a nested sequence of simplexes $\Delta_{i_n}^n$ such that $\tau(\sigma\circ\partial_0\Delta_{i_n}^n)\neq e$. Let $x=\bigcap_{n=0}^{\infty}\Delta_{i_n}^n$. Let $\rho_s(t)$ be a homotopy of loops at x such that $\rho_{1/n}=\partial_x\Delta_{i_n}^n$. Then by Lemma 1 there exists $s_0>0$ such that $\tau(\sigma\circ\rho_s)=e$ for $s\leqq s_0$; in particular there is some n such that $\tau(\sigma\circ\partial_x\Delta_{i_n}^n)=e$. But this implies $\tau(\sigma\circ\partial_0\Delta_{i_n}^n)=e$ which is the desired contradiction. q.e.d.

Theorem. *A bundle over a finite polyhedron with a structural group G, which has no small subgroups, has a transitive translation function if and only if it is equivalent in G to an H bundle where H is a totally disconnected subgroup of G.*

Proof. Let τ be the translation function that exists by the Hurewicz uniformization theorem for the H bundle. Then since the maps $\tau(\omega_1\cdot\omega_2)$ and $\tau(\omega_2)\circ\tau(\omega_1)$ are homotopic and H is totally disconnected it follows that they are equal. Hence τ is transitive.

If, on the other hand, there is a transitive translation function τ, we can construct a coordinate bundle with a totally disconnected group. Select a point b_0 in the base space and let $\phi_0(b_0)$: $Y\rightarrow p^{-1}(b_0)$ be one of the coordinate maps restricted to $Y\times b_0$. For every vertex a_i select a contraction of its star to the point b_0. Thus for every $x\in\mathrm{St}\,a_i$ we obtain a path $\omega_{a_i,x}$ from b_0 to x. Define the maps ϕ_{a_i}: $Y\times\mathrm{St}\,a_i\rightarrow p^{-1}(\mathrm{St}\,a_i)$ by $\phi_{a_i}(y,x)=\tau(\phi_0(y),\omega_{a_i,x})$. Thus we obtain coordinate maps which are compatible with the original coordinate maps. The subgroup H of G which is spanned by the $g_{a_i,a_j}(x)$, $x\in\mathrm{St}\,a_i\cap\mathrm{St}\,a_j$, is the continuous image of the finitely generated fundamental group of the base space and hence H is totally disconnected. q.e.d.

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