A CONTINUOUS EXACT SET[1]

LEONARD GILLMAN

A linearly ordered set $\mathfrak{A}$ is called *exact*[2] if it is not similar to any of its proper subsets. It is easily seen that $\mathfrak{A}$ is exact if and only if the only similarity transformation of $\mathfrak{A}$ into itself is the identity [**9**, p. 341]. The first example of an exact set was given by Dushnik and Miller in [**5**]; for related investigations, such as the decomposition of a set into exact subsets, see the bibliography below. The standard construction is to delete suitably chosen elements from a contionuus set (e.g., the reals), with the result that the exact set so obtained has gaps. Cuesta [**2**] has asked whether there can exist a *continuous* exact set.[3]

In this paper, we construct a continuous exact set of power $\mathfrak{c} = 2^{\aleph_0}$. The proof of exactness, however, requires the arithmetical hypothesis that $\mathfrak{c}$ is a *regular* cardinal number, that is, that the sum of fewer than $\mathfrak{c}$ cardinals, each of which is less than $\mathfrak{c}$, is itself less than $\mathfrak{c}$. (In particular, this will be the case if the continuum hypothesis is true.)

1. **The family** $\mathfrak{F}$. The closed interval $[0, 1]$ of the reals will be denoted throughout by L. From results of Ginsburg, e.g., [**9**, Theorem 2.2], one obtains an unbordered subset E of L such that every interval of L contains $\mathfrak{c}$ elements of E, and such that no two disjoint subsets of E of power $\mathfrak{c}$ are similar. Let $\mathfrak{F}$ be a family of $\mathfrak{c}$ mutually disjoint subsets of E, such that, for every $F \in \mathfrak{F}$,

(1.1) *every interval of* L *contains* $\mathfrak{c}$ *elements of* F.

To obtain such a decomposition is not difficult.

Since distinct elements of $\mathfrak{F}$ are disjoint, we have

(1.2) *for distinct elements* F, F' *of* $\mathfrak{F}$, *no subset of* F *of power* $\mathfrak{c}$ *is similar to any subset of* F'.

Since each $F \in \mathfrak{F}$ is unbordered, we have

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[2] The term is due to S. Ginsburg.

[3] I am indebted to F. Bagemihl for calling this to my attention.

(1.3) $$0 \notin F, \quad 1 \notin F \qquad (F \in \mathfrak{F}).$$

The following lemma is easily generalized, but we state here only what we shall need for application below.

LEMMA 1. *Let* $|U| < \mathfrak{c}$, *and let* S, F_s $(s \in S)$, *and* F'_u $(u \in U)$ *be mutually disjoint subsets of elements of* $\mathfrak{F}$. *Then, if* $\mathfrak{c}$ *is a regular cardinal, no subset of power* $\mathfrak{c}$ *of the set* $T = \sum_{s \in S} F_s$ *is similar to the set* $W = \sum_{u \in U} F'_u$.

PROOF. Let $A \subset T$ be of power $\mathfrak{c}$, and suppose that there exists a similarity transformation f of A to W. If there is an $s \in S$ such that $|A \cap F_s| = \mathfrak{c}$, choose any such s, and define $B = A \cap F_s$. If there is no such s, then the regularity of $\mathfrak{c}$ implies that A meets $\mathfrak{c}$ different sets F_s; in this case, A has a subset of power $\mathfrak{c}$ that is imbeddable in S, and we define B to be any such subset. Since $|U| < \mathfrak{c}$, and $\mathfrak{c}$ is regular, there exist a $u \in U$, and a subset C of B of power $\mathfrak{c}$, such that $f(C) \subset F'_u$. This, however, contradicts (1.2).

2. **The set** $\mathfrak{A}$. We write $\mathfrak{F} = (F_{n,x})$, with the indices chosen inductively as follows (distinct indices denoting distinct elements of $\mathfrak{F}$):
for $n = 1$, *we have* $x = 0$;
for $n > 1$, *x ranges over the set* $\bigcup_y F_{n-1,y}$.
(This scheme may be accomplished by first decomposing the family $\mathfrak{F} - \{F_{1,0}\}$ into $\aleph_0$ subfamilies, each of power $\mathfrak{c}$.) Thus if $F_{m,y}$ exists (i.e., if this indexing occurs according to the above scheme), then $F_{m+1,x}$ exists for each $x \in F_{m,y}$. For every $F \in \mathfrak{F}$, each $x \in F$ appears just once as a second index; to each first index $n > 1$, there correspond $\mathfrak{c}$ second indices x; and $F_{n,x} = F_{m,y}$ if and only if $n = m$ and $x = y$.

The notation $x \notin F_{m,y}$ will be understood to include the case that $F_{m,y}$ does not exist. With this convention, we define $\mathfrak{A}$ to be the set of all sequences $\xi = (x_i)_{0<i<\omega}$ of elements of L, with coordinates x_i restricted inductively as follows. Define $x_0 = 0$. Then

(2.1)
$(x_i) \in \mathfrak{A}$ *if and only if* $x_1 \in L$, *and for each* $i > 0$,
(a) *if* $x_i \in F_{i,x_{i-1}}$, *then* $x_{i+1} \in L$,
(b) *if* $x_i \notin F_{i,x_{i-1}}$, *then* $x_{i+1} = 0$.

The set $\mathfrak{A}$ will be our continuous exact set.

From (2.1b), if $n > 0$ and $x_{n+1} \neq 0$, then $x_n \in F_{n,x_{n-1}}$, whence, by the indexing scheme, F_{n+1,x_n} exists. For $n = 0$, $F_{n+1,x_n} = F_{1,0}$ exists in any case. So we have:

(2.2) *if* $(x_i) \in \mathfrak{A}$, *and* $x_{n+1} \neq 0$, *then* F_{n+1,x_n} *exists* $\qquad (n = 0, 1, \cdots)$.

Since $0 \notin F$ $(F \in \mathfrak{F})$, iteration of (2.1b) yields

(2.3) *if* $(x_i) \in \mathfrak{A}$, *and* $x_{n+1} \notin F_{n+1,x_n}$, *then* $x_j = 0$ *for all* $j > n+1$ $(n = 0, 1, \cdots)$.

$\mathfrak{A}$ has a first element, namely $(0, 0, \cdots)$. Since $1 \notin F_{1,0}$, $\mathfrak{A}$ has a last element, namely, $(1, 0, 0, \cdots)$; we denote this latter by π.

For each $\xi = (x_i) \in \mathfrak{A}$, we define the finite sequences

$$(2.4) \qquad \begin{aligned} \phi_n(\xi) &= (x_1, \cdots, x_n) \qquad (n = 1, 2, \cdots), \\ \phi_{-1}(\xi) &= \phi_0(\xi) = \varnothing. \end{aligned}$$

LEMMA 2. *Let* $\xi = (x_i) \in \mathfrak{A}$ *and* $\eta = (y_i) \in \mathfrak{A}$, *and consider any fixed* $n = 1, 2, \cdots$. *If* $x_{n+1} \neq 0$, $y_{n+1} \neq 0$, *and* F_{n+1,x_n} *meets* F_{n+1,y_n} (*cf.* (2.2)) *then* $\phi_n(\xi) = \phi_n(\eta)$.

PROOF. Since distinct elements of $\mathfrak{F}$ are disjoint, we have $x_n = y_n$, whence the result holds for $n = 1$. Since $x_{n+1} \neq 0$, we have $x_n \in F_{n,x_{n-1}}$, by (2.1b), whence $x_n \neq 0$, by (1.3). Likewise, $y_n \in F_{n,y_{n-1}}$, and the result now follows by induction.

3. **Proof that $\mathfrak{A}$ is continuous.** Let $\mathfrak{P}$ be any nonempty subset of $\mathfrak{A}$. Define $L_0 = \{0\}$. Consider any $m < \omega$, and suppose that a subset L_i of L has been defined for all $i \leqq m$, and that L_i has a last element y_i for all $i \leqq m$. If L_m has a last element y_m, define

$$L_{m+1} = \{x_{m+1} \colon (x_i) \in \mathfrak{P},\ x_i = y_i \textit{ for all } i \leqq m\}.$$

If L_i is thus defined for all $i < \omega$, then, clearly, $(y_i)_{0<i<\omega}$ is the least upper bound of $\mathfrak{P}$ in $\mathfrak{A}$.

In the contrary case, let m denote the smallest integer such that L_{m+1} is not defined, i.e., such that L_m has no last element. Denote the least upper bound of L_m in L by z_m. Define $z_i = y_i$ for all $i < m$, and $z_i = 0$ for all $i > m$. Then, evidently, $(z_i)_{0<i<\omega}$ is the least upper bound of $\mathfrak{P}$ in $\mathfrak{A}$.

Thus every nonempty subset of $\mathfrak{A}$ has a least upper bound in $\mathfrak{A}$. Obviously, $\mathfrak{A}$ is dense. Hence $\mathfrak{A}$ is continuous.

4. **The set $\mathfrak{T}$.** The remainder of our paper is devoted to the proof that $\mathfrak{A}$ is exact. We suppose that, on the contrary, there exists a similarity transformation f of $\mathfrak{A}$ into $\mathfrak{A}$ that is not the identity. Let $\alpha = (a_i) \in \mathfrak{A}$ satisfy $f(\alpha) \neq \alpha$, and write $f(\alpha) = \beta = (b_i)$. We shall assume that $\alpha < \beta$, the argument in the opposite case being analogous to the one that follows. Then f maps the interval $]\alpha, \beta[$ of $\mathfrak{A}$ into the interval $]\beta, \pi[$ of $\mathfrak{A}$.

Let m denote the first index at which α and β differ: $a_i = b_i$ for all

$i<m$, and $a_m<b_m$. Then $b_m\neq 0$, so F_{m,b_m-1} exists, by (2.2). By (1.1), every interval of L contains $\mathfrak{c}$ elements of F_{m,b_m-1}. Hence the set

$$S =]a_m, b_m[\cup F_{m,b_m-1} \tag{4.1}$$

is of power $\mathfrak{c}$. Evidently, $F_{m+1,s}$ exists for every $s\in S$. Therefore the set

$$T = \sum_{s\in S} F_{m+1,s} \tag{4.2}$$

has $\mathfrak{c}$ mutually disjoint intervals.

Let $\mathfrak{H}$ denote the set of all elemenets $\theta=(h_i)\in\mathfrak{A}$ such that $h_i=b_i$ for all $i<m$, $h_m\in S$, $h_{m+1}\in F_{m+1,h_m}$, $h_{m+2}\in L$, and $h_i=0$ for all $i>m+2$. Then for all $\theta\in\mathfrak{H}$, we have $a_m<h_m<b_m$, whence $\alpha<\theta<\beta$. Thus, $\mathfrak{H}\subset]\alpha,\beta[$.

Since f maps $]\alpha,\beta[$ into $]\beta,\pi[$, there is a subset $\mathfrak{K}$ of $]\beta,\pi[$ that is similar to $\mathfrak{H}$. Now, clearly, $\mathfrak{H}$ is similar to LT. Consequently, we may write

$$\mathfrak{K} = \sum_{\tau\in\mathfrak{T}} \mathfrak{L}_\tau, \tag{4.3}$$

where

$$\mathfrak{T}\simeq T, \tag{4.4}$$

and $\mathfrak{L}_\tau\simeq L$ for all $\tau\in\mathfrak{T}$. Furthermore, we may suppose that τ is the first element of $\mathfrak{L}_\tau$ $(\tau\in\mathfrak{T})$, so that we have $\mathfrak{T}\subset\mathfrak{K}$. Obviously, $\beta<\kappa$ for all $\kappa\in\mathfrak{K}$; in particular, then,

$$\beta < \tau \textit{ for all } \tau\in\mathfrak{T}. \tag{4.5}$$

We now define

$$\mathfrak{R}_n = \{\phi_n(\tau) : \tau\in\mathfrak{T}\} \qquad (n = -1, 0, 1, 2, \cdots), \tag{4.6}$$

where ϕ_n is as in (2.4) (thus, $\mathfrak{R}_{-1}=\mathfrak{R}_0=\{\varnothing\}$); and

$$\mathfrak{T}_{n,\rho} = \{(t_n, t_{n+1}) : \tau = (t_i)\in\mathfrak{T}, \phi_{n-1}(\tau) = \rho\} \qquad (\rho\in\mathfrak{R}_{n-1}, n = 0, 1, \cdots). \tag{4.7}$$

Obviously, for each $\rho\in\mathfrak{R}_{n-1}$, $\mathfrak{T}_{n,\rho}$ is a subset of the set

$$\mathfrak{A}_{n,\rho} = \{(x_n, x_{n+1}) : \xi = (x_i)\in\mathfrak{A}, \phi_{n-1}(\xi) = \rho\}. \tag{4.8}$$

5. **The sets T_n.** Next, we define

$$T_n = \{t_n : \tau = (t_i)\in\mathfrak{T}\} \qquad (n = 0, 1, \cdots), \quad T_{-1} = \{\varnothing\}. \tag{5.1}$$

(Note that $T_0=\{0\}$.) Most of the rest of the paper is concerned with the proof of the following

LEMMA. $|T_i|<\mathfrak{c}$ *for all* i.

The proof is by induction. Trivially, $|T_{-1}|=|T_0|<\mathfrak{c}$. Fix $n\geqq 0$, and suppose that $|T_i|<\mathfrak{c}$ for all $i\leqq n$. Then

$$(5.2) \qquad |T_n| < \mathfrak{c};$$

also, referring to (4.6), we see that

$$(5.3) \qquad |\mathfrak{R}_{n-1}| \leqq |T_{-1}|\cdot|T_0|\cdot\cdots\cdot|T_{n-1}| < \mathfrak{c}.$$

Now fix $\rho\in\mathfrak{R}_{n-1}$, and consider the set $\mathfrak{T}_{n,\rho}$ (4.7). For each $(u, v)\in\mathfrak{T}_{n,\rho}$, choose one element $\tau(u, v)=(t_i)\in\mathfrak{T}$ such that

$$(5.4) \qquad \phi_{n-1}(\tau(u, v)) = \rho, \qquad t_n = u, \qquad t_{n+1} = v,$$

and denote the set of these elements $\tau(u, v)$ by $\mathfrak{T}'$:

$$(5.5) \qquad \mathfrak{T}' = \{\tau(u, v): (u, v) \in \mathfrak{T}_{n,\rho}\}.$$

Obviously,

$$(5.6) \qquad \mathfrak{T}' \subset \mathfrak{T}, \qquad \mathfrak{T}' \simeq \mathfrak{T}_{n,\rho}$$

(with the lexicographic order on the pairs $(u, v)\in\mathfrak{T}_{n,\rho}$).

Now define

$$(5.7) \qquad \mathfrak{K}' = \sum_{\tau\in\mathfrak{T}'} \mathfrak{L}_\tau.$$

Then $\mathfrak{K}'\subset\mathfrak{K}$ (4.3).

Next, we decompose $\mathfrak{T}_{n,\rho}$ into the complementary subsets

$$(5.8) \qquad \begin{aligned} \mathfrak{V} &= \{(u, v) \in \mathfrak{T}_{n,\rho}: v \notin F_{n+1,u}\}, \\ \mathfrak{W} &= \{(u, v) \in \mathfrak{T}_{n,\rho}: v \in F_{n+1,u}\} \end{aligned}$$

(in case $F_{n+1,u}$ does not exist, then $\mathfrak{W}$ is empty), which we proceed to investigate separately.

6. **Proof that** $|\mathfrak{V}|<\mathfrak{c}$. Consider any $(u, v)\in\mathfrak{V}$. If $\sigma=(s_i)$ is any element of $\mathfrak{A}$ such that $\phi_{n-1}(\sigma)=\rho$, $s_n=u$, and $s_{n+1}=v$, then, since $v\notin F_{n+1,u}$ (5.8), we have from (2.3) that $s_i=0$ for all $i>n+1$. Hence $\tau(u, v)$ is the only such element $\sigma\in\mathfrak{A}$ (5.4). It follows that if (u, v) is a left-hand limit element of $\mathfrak{T}_{n,\rho}$ in the set $\mathfrak{A}_{n,\rho}$ (4.8), then $\tau(u, v)$ is a left-hand limit element of $\mathfrak{T}'$ (5.5)—hence of $\mathfrak{K}'$ (5.7)—in $\mathfrak{A}$. But then the set $\mathfrak{L}_{\tau(u,v)}$ (5.7) cannot exist.

Therefore (u, v) is not a left-hand limit element of $\mathfrak{T}_{n,\rho}$ in $\mathfrak{A}_{n,\rho}$. There accordingly exists an interval

$$\mathfrak{J}(u, v) =](u, v), (u', v')[$$

of $\mathfrak{A}_{n,\rho}$ that is free of elements of $\mathfrak{T}_{n,\rho}$. Now either $u<u'$, or $u=u'$ and $v<v'$. Since $u\in T_n$ (5.1), and $|T_n|<\mathfrak{c}$ (5.2), there are less than $\mathfrak{c}$ intervals $\mathfrak{J}(u, v)$ with $u<u'$. On the other hand, if $u=u'$, then $\mathfrak{J}(u, v)$ is similar to the interval $]v, v'[$ of L; so for each $u\in T_n$, there are (at most) denumerably many intervals $\mathfrak{J}(u, v)$ with $u=u'$. Consequently, there are less than $\mathfrak{c}$ intervals $\mathfrak{J}(u, v)$ altogether. It follows that $|\mathfrak{V}|<\mathfrak{c}$.

7. **Proof that** $|\mathfrak{W}|<\mathfrak{c}$. Since $\mathfrak{W}\subset\mathfrak{T}_{n,\rho}$ (5.8), $\mathfrak{T}_{n,\rho}\simeq\mathfrak{T}'$ (5.6), $\mathfrak{T}'\subset\mathfrak{T}$ (5.6), and $\mathfrak{T}\simeq T$ (4.4), we see that T contains a subset that is similar to $\mathfrak{W}$. Define

$$(7.1)\qquad \begin{aligned} U &= \{u \in T_n\colon \textit{there exists } v \textit{ such that } (u, v) \in \mathfrak{W}\}, \\ F_u' &= \{v \in F_{n+1,u}\colon (u, v) \in \mathfrak{W}\} \qquad (u \in U). \end{aligned}$$

Then $F_u'\subset F_{n+1,u}$ $(u\in U)$. Referring to (5.8), we see that

$$(7.2)\qquad \mathfrak{W} \simeq W, \quad \text{where} \quad W = \sum_{u\in U} F_u'.$$

Thus, T has a subset that is similar to W.

Now, according to the indexing of the family $\mathfrak{F}$, the sets F_u' $(u\in U)$ are mutually disjoint. Likewise, the sets $F_{m+1,s}$ $(s\in S)$ that appear in the definition of T (4.2, 4.1) are mutually disjoint, and each is disjoint from S. Furthermore, if $n<m-1$, or $n>m$, then every F_u' is disjoint from S and from every $F_{m+1,s}$. Also, every F_u' is disjoint from every $F_{m+1,s}$ in case $n=m-1$, and from S in case $n=m$.

Suppose that $n=m-1$, and that there is a $u\in U$ such that $F_{n+1,u}$ meets S. Now $S\subset F_{m,b_{m-1}}$ (4.1); hence $F_{n+1,u}$ meets $F_{m,b_{m-1}}$. Consider any $v\in F_u'$. Then for $(t_i)=\tau(u, v)$, we have $t_n=u$, and $t_{n+1}=v\neq 0$. Since $b_m>a_m$, we have $b_m\neq 0$. It follows from Lemma 2 that $\phi_n(\tau(u, v))=\phi_n(\beta)$. Now $\beta<\tau(u, v)$ (4.5). Therefore we must have $b_{n+1}\leqq t_{n+1}$, i.e., $b_m\leqq v$. Since v was an arbitrary element of F_u', we conclude that $F_u'\subset[b_m, 1]$. But $S\subset]a_m, b_m[$ (4.1). Consequently, F_u' does not meet S. It follows that for every $u\in U$, F_u' is disjoint from S.

Next, suppose that $n=m$, and that there exist $u\in U$ and $s\in S$ such that $F_{n+1,u}$ meets $F_{m+1,s}$. Consider any $v\in F_u'$. Then for $(t_i)=\tau(u, v)$, we have $t_n=u$, and $t_{n+1}=v\neq 0$. Since $s\in F_{m,b_{m-1}}$, there is a $\sigma=(s_i)\in\mathfrak{A}$ such that $\phi_m(\sigma)=(b_1, \cdots, b_{m-1}, s)$, and $s_{m+1}\neq 0$ (2.1a). By Lemma 2, we have $\phi_n(\tau(u, v))=\phi_m(\sigma)$. But then, since $\phi_m(\beta)=(b_1, \cdots, b_{m-1}, b_m)$, and $s<b_m$ (4.1), we find that $\tau(u, v)<\beta$, contradicting (4.5). It follows that every F_u' must be disjoint from every $F_{m+1,s}$.

Thus, in all cases, the sets S, $F_{m+1,s}$ $(s\in S)$, and F'_u $(u\in U)$ are mutually disjoint. Finally, we have $|U|<\mathfrak{c}$, since $U\subset T_n$ (7.1) and $|T_n|<\mathfrak{c}$ (5.2). Since T (4.2) has a subset that is similar to W (7.2 ff.), we conclude from Lemma 1 (assuming that $\mathfrak{c}$ is regular) that $|W|$ must be less than $\mathfrak{c}$. Therefore $|\mathfrak{W}|<\mathfrak{c}$ (7.2).

8. **Conclusion.** Thus, $|\mathfrak{V}|<\mathfrak{c}$ and $|\mathfrak{W}|<\mathfrak{c}$. Hence $|\mathfrak{T}_{n,\rho}|<\mathfrak{c}$ (5.8). Define

$$\mathfrak{T}_n = \{(t_n, t_{n+1}) : \tau = (t_i) \in \mathfrak{T}\}. \tag{8.1}$$

Then $\mathfrak{T}_n=\bigcup_{\rho\in\mathfrak{R}_{n-1}}\mathfrak{T}_{n,\rho}$ (4.7). Now $|\mathfrak{T}_{n,\rho}|<\mathfrak{c}$ for each $\rho\in\mathfrak{R}_{n-1}$, as has just been shown, and $|\mathfrak{R}_{n-1}|<\mathfrak{c}$, by (5.3). It follows (since $\mathfrak{c}$ is regular) that $|\mathfrak{T}_n|<\mathfrak{c}$. Now from (5.1) and (8.1), it is clear that $|T_{n+1}|\leqq|\mathfrak{T}_n|$. Hence $|T_{n+1}|<\mathfrak{c}$. This is our induction step, and the proof of the lemma is now complete.

Finally, define $M=\bigcup_{0<n<\omega} T_n$. Then[4] $|M|<\mathfrak{c}$. We may as well suppose that M contains all the rational numbers in L. Let $\mathfrak{M}$ denote the set of all $\xi=(x_i)\in\mathfrak{A}$ for which every $x_i\in M$. Obviously, $\mathfrak{T}\subset\mathfrak{M}$ (5.1). Now $\mathfrak{M}$ has a dense subset of power $|M|<\mathfrak{c}$, namely, the set of all $\xi\in\mathfrak{M}$ for which only finitely many $x_i\neq 0$. On the other hand, $\mathfrak{T}\simeq T$ (4.4), whence $\mathfrak{T}$ has $\mathfrak{c}$ mutually disjoint intervals (4.2 ff.). This contradiction completes the proof that $\mathfrak{A}$ is an exact set.

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Purdue University

[4] This follows from König's theorem, whether $\mathfrak{c}$ is regular or not.