Corollary 2. If $\phi: H^k(A) \to H^k(\overline{A})$ is an isomorphism for $k \leq n$ and univalent for $k = n+1$, and $\phi: H(H^+(X, A)) \to H(H^+(X, \overline{A}))$ is an isomorphism, then $\phi: H^k(TA) \to H^k(T\overline{A})$ is also an isomorphism for $k \leq n$ and univalent for $k = n+1$.

Reference


University of California, Los Angeles

---

**ONE-PARAMETER TRANSFORMATION GROUPS IN THE PLANE**

**PAUL S. MOSTERT**

Very little is known about the action of a one-parameter group $R$ on two-space except when all orbits are circles, in which case the action is completely known [1]. In a forthcoming paper, A. Beck proves that any closed set can act as the set of fixed points for $R$. Hence, a very general description appears to be hopeless. However, here, we are able to prove the following result.

**Theorem.** Let $E$ be the plane and $R$ the real line acting on $E$ as a group of transformations without fixed points (i.e., no point is left fixed by all of $R$). If $E/R$ is Hausdorff, then $E$ is fibred as a direct product of $R$ and a cross sectioning line. Thus, $R$ is equivalent to a group of translations.

**Proof.** Let $x \in E$. Since $x$ is not fixed under $R$, there is a closed interval $[-a, a] = T$ about 0 in $R$, and an arc $C \subseteq E$, $x \in C$ but not an end point of $C$, such that $T^2(C)$ is a compact neighborhood of $x$ and the mapping $(t, c) \to t(c)$ is one to one from $T^2 \times C \to T^2(C)$. That is, $C$ is a local cross section to the local orbits of $T^2$ [1]. We shall show that $C$ is a local cross section for the orbits of $R$.

Suppose, on the contrary, that for some $z \in C$, there is an $r > a$ such that $r(z) \in T(C)$. Let $b$ be the greatest lower bound of such numbers. Then $b(z) \subseteq -a(C)$, for if not, say $b(z) = t(c)$, $t \in T$, $c \in C$, and $t > -a$, then there is a $t'$, $-a < t' < 0$, such that $t + t' > -a$. Hence $(b + t')(z) = (t + t')(c) \in T(C)$. But $t' < 0$ so that $b + t' < b$. By the choice of $b$, this implies $b + t' < a$. Since this implies $b + t'' < a$ for

Presented to the Society, April 19, 1957; received by the editors December 21, 1956.
that \( t' \leq t < 0, b = a \). That is, for some \( t, 0 < t < a \), \((b + t)(z) \in T(c)\). But this contradicts the fact that \( T^2(C) \) is homeomorphic to \( T^2 \times C \). Thus \( b > a \), and \( b(z) \in -a(C) \).

Suppose \( b(z) = -a(z) \). Then \([ -a, b ](z) = R(z) \) is a circle bounding a pre-compact region \( A \) by the Jordan curve theorem. Hence, if \( z \in A, R(z) \subset A \) since orbits cannot intersect. Then, for any \( r \in R, r(A^-) \subset A^- \). Thus \( r \) has a fixed point. For each \( n \), let \( x_n \) be a fixed point for \( 1/2^n \). Let \( y \) be a limit point of the \( x_n \)'s. Thus \( R(y) = y \), contradicting our hypothesis that \( R \) acts without fixed points.

Now we may assume \( b(z) \neq -a(z) \). Let \([b(z), -a(z)] = -a[(a + b)(z), z] \) denote the arc of \(-a(C)\) joining \( b(z) \) to \(-a(z)\). Then \([-a, b](z) \cup [b(z), -a(z)] \) is a simple closed curve in \( E \), and thus divides \( E \) into two parts \( A \) and \( B \) one of which is pre-compact. Moreover, if \( R_+ = (0, \infty), R_- = (-\infty, 0), R_+(b(z)) \) is contained in one, say it is \( A \), while \( R_-(a(z)) \) is contained in the other, for \( R_+(b(z)) \) cannot leave \( A \) except by crossing \([b(z), -a(z)] \) since an orbit cannot cross itself. But \( R_+(b(z)) \) cannot leave \( T(C) \) except by crossing \( a(C) \). A similar argument establishes that \( R_-(a(z)) \subset B \). Moreover, if \( z \in A \), and \( r > 0 \), then \( r(z) \in A \) since again \( R(z) \) cannot cross \([-a, b](z) \), and cannot leave \( T(C) \) through \(-a(C)\). Hence, for \( r > 0, r(A^-) \subset A^- \). But \( A^- \) is a closed two-cell since it is bounded by a simple closed curve. Thus \( r \) has a fixed point. Again find \( x_n \in A^- \) such that \( 1/2^n(x_n) = x_n \). If \( y \) is a limit point of \( x_n \)'s, then \( R(y) = y \), a contradiction.

This proves that \( C \) meets \( R(x) \) in exactly one point for each \( x \in T(C) \). Thus, \( C \) is a local cross section for all of \( R \). Since we can find a local cross section for each point of \( E \), and since \( E/R \) is Hausdorff, \( E \) is a fibre bundle over \( E/R \) with fibre \( R \) and base space a connected one-dimensional manifold. Since \( E \) is a principal fibre bundle over \( E/R \), and \( R \) is the line, there is a cross section \( L \) of \( E/R \) in \( E \) such that the natural mapping \( E \times R \to R(E) \) is a homeomorphism onto \( E \) \([2]\). Clearly \( L \) must be a line.

**Bibliography**


Tulane University