A NOTE ON THE GAUSS-GREEN THEOREM

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A very general measure-theoretic version of the Gauss-Green formula (for $n$-space) has recently been studied by De Giorgi in [DG 1] and [DG 2]. Independently Fleming and Young have obtained related results (for 3-space) in [FY], employing the technique of “generalized surfaces.” Here the work of De Giorgi will be supplemented through use of the geometric concept of exterior normal introduced in [F 1].

Assuming that $A$ is a subset of Euclidean $n$-space $E_n$, measurable with respect to Lebesgue measure $L_n$, consider the following two conditions:

1. There exist finite real valued (signed) Borel measures $\Phi_1, \cdots, \Phi_n$ over $E_n$ such that

$$\int_A D_ifdL_n = \int_{E_n} f d\Phi_i \quad \text{for } i = 1, \cdots, n$$

whenever $f$ is a continuously differentiable function on $E_n$ which vanishes at infinity. [In the Schwartz language of distributions this means that the partial derivatives of the characteristic function of $A$ are the measures $-\Phi_1, \cdots, -\Phi_n$.]

2. There exists a number $M$ and an infinite sequence of sets $A_j$ with polyhedral (or smooth) boundaries $B_j$ such that

$$L_n[(A - A_j) \cup (A_j - A)] \to 0 \text{ as } j \to \infty$$

and

$$H^{n-1}_n(B_j) \leq M \text{ for all positive integers } j,$$

where $H^{n-1}_n$ is $n-1$ dimensional Hausdorff measure over $E_n$.

De Giorgi proved in [DG 1] that the conditions (1) and (2) are equivalent. He also showed that the total variation over $E_n$ of the vector-valued measure $\Phi$ defined by $\Phi_1, \cdots, \Phi_n$ in (1) is equal to the infimum of all numbers $M$ suitable for (2). In his subsequent paper [DG 2], he established various interesting local properties of $\Phi$, which will be used and extended presently.

Say, as in [F 1], that a unit vector $u$ is an exterior normal of $A$ at $x$ if and only if

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\( r^{-n}L_n(\{y \mid |y - x| < r, (y - x) \cdot u < 0, y \in A\}) \rightarrow 0 \)

and

\( r^{-n}L_n(\{y \mid |y - x| < r, (y - x) \cdot u > 0, y \in A\}) \rightarrow 0 \)
as \( r \to 0^+ \), where \( \cdot \) is the inner product. Such a unit vector \( u \), if it exists, is uniquely determined by \( A \) and \( x \), and denoted \( \nu(A, x) \). In case no such \( u \) exists, \( \nu(A, x) \) is the null vector. This defines for each \( x \in E_n \) a vector \( \nu(A, x) \) with components \( \nu_1(A, x), \ldots, \nu_n(A, x) \).

It will be shown that (1) and (2) are equivalent to the condition:

\[
(3) \quad H_n^{n-1}(\{x \mid \nu(A, x) = 1\}) < \infty
\]

and

\[
\int_A D_i f dL_n = \int_{E_n} f(x)\nu_i(A, x) dH_n^{n-1} x \quad \text{for } i = 1, \ldots, n,
\]

whenever \( f \) is a continuously differentiable function on \( E_n \) which vanishes at infinity.

Obviously (3) implies (1) with

\[
\Phi_i(S) = \int_S \nu_i(A, x) dH_n^{n-1} x
\]

whenver \( S \) is a Borel set contained in \( E_n \).

The fact that (1) implies (3) will be established with the help of the following lemma:

**If (1) holds and if \( |\nu(A, x)| = 1 \), then**

\[
\limsup_{r \to 0^+} r^{-n+1} \left\| \Phi[K(x, r)] \right\| \geq \frac{\alpha(n)}{4},
\]

where \( K(x, r) = \{y \mid |y - x| < r\} \) and \( \alpha(n) = L_n[K(x, 1)] \).

In the proof of this lemma one may assume, to simplify the notation, that \( x \) is the origin and \( \nu(A, x) \) is the \( n \)th base vector \((0, \ldots, 0, 1)\). According to \([DG 2, \text{Lemma III}]\) it is true for \( L_1 \) almost all positive numbers \( r \) that

\[
\int_{A \cap K(x, r)} D_n f dL_n = \int_{K(x, r)} f d\Phi_n + \int_A f(y)\nu_n[K(x, r), y] dH_n^{n-1} y
\]

whenever \( f \) is a continuously differentiable function on \( E_n \) which vanishes at infinity. Moreover, since \( A \cap K(x, r) \) is bounded, vanishing at infinity is irrelevant; and continuously differentiable may be generalized to Lipschitzian, by smoothing. In particular the formula applies with
\[ f(y) = y_n + \left| r^2 - \sum_{i=1}^{n-1} (y_i)^2 \right|^{1/2} \quad \text{for } y \in E_n. \]

Consider the hemispherical shells
\[ V(r) = \{ y \mid |y| = r \text{ and } y_n \leq 0 \}, \]
\[ W(r) = \{ y \mid |y| = r \text{ and } y_n > 0 \}, \]
and note that
\[ f(y) = 0 \text{ for } y \in V(r), \quad |f(y)| \leq 2r \text{ for } y \in K(x, r), \]
\[ L_n[A \cap \{ y \mid |y| < s \text{ and } y_n > 0 \}] = \int_0^s H_n^{n-1}[A \cap W(r)]dr \quad \text{for } s > 0. \]

Now let \( \epsilon > 0 \). Inasmuch as
\[ L_n[A \cap \{ y \mid |y| < s \text{ and } y_n > 0 \}] \leq \epsilon \frac{s^n}{n} = \int_0^s \epsilon r^{n-1}dr \]
for small \( s > 0 \), there exist arbitrarily small \( r > 0 \) for which
\[ H_n^{n-1}[A \cap W(r)] \leq \epsilon r^{n-1}, \]
which are unexceptional with respect to [DG 2, Lemma III], and for which
\[ L_n[A \cap K(x, r)] \geq \frac{r^n}{2} [\alpha(n) - \epsilon]. \]

For such \( r \) it follows that
\[ \int_{A \cap K(x, r)} D_n f dL_n = L_n[A \cap K(x, r)] \geq \frac{r^n}{2} [\alpha(n) - \epsilon], \]
\[ \int_{K(x, r)} f d\Phi_n \leq 2r \left| \Phi_n[K(x, r)] \right| \leq 2r \left| \Phi[K(x, r)] \right|, \]
\[ \int_A f(y) \nu_n[K(x, r), y]dH_n^{n-1}y \leq 2r H_n^{n-1}[A \cap W(r)] \leq 2\epsilon r^n, \]
hence
\[ r^{-n+1} \left| \Phi[K(x, r)] \right| \geq \frac{1}{4} [\alpha(n) - \epsilon] - \epsilon = \frac{1}{4} [\alpha(n) - 5\epsilon]. \]

Now suppose (1) holds and let \( \mu \) be the total variation measure associated with the vector valued measure \( \Phi \). The preceding lemma implies that the \( n-1 \) dimensional \( \mu \) density of \( E_n \) is no less than
\( \alpha(n)/[4\alpha(n-1)] \) at each point of the set

\[ N = \{ x \mid v(A, x) = 1 \}. \]

If \( S \) is any Borel set contained in \( N \), then \( \mu(E_n - S) < \infty \), and it follows from \([F \, 3, \, 3.3 \, \text{and} \, 3.1]\), that

\[ H_n^{n-1}(S) \leq \frac{\alpha(n)}{4\alpha(n - 1)} \mu(S). \]

Furthermore De Giorgi established in \([DG \, 2, \, \text{Theorems} \, \text{III} \, \text{and} \, \text{IV}]\) the existence of a Borel set \( F \) such that

\[ \mu(E_n - F) = 0, \quad F \subset N \]

and

\[ \mu(S) = H_n^{n-1}(S), \quad \Phi_i(S) = \int_S v_i(A, x) dH_n^{n-1} x \quad \text{for} \quad i = 1, \cdots, n \]

whenever \( S \) is any Borel set contained in \( F \). Inasmuch as

\[ H_n^{n-1}(N - F) \leq \frac{\alpha(n)}{4\alpha(n - 1)} \mu(N - F) = 0, \]

it follows that if \( S \) is any Borel set contained in \( E_n \), then

\[ \Phi_i(S) = \Phi_i(S \cap F) = \int_{S \cap F} v_i(A, x) dH_n^{n-1} x = \int_{S \cap N} v_i(A, x) dH_n^{n-1} x = \int_S v_i(A, x) dH_n^{n-1} x \]

for \( i = 1, \cdots, n \). Accordingly (3) is a consequence of (1).

In the special case when \( H_n^{n-1}(\text{Boundary } A) < \infty \), the condition (2) is obviously satisfied. Therefore (3) holds in this case, as shown by quite different methods in \([F \, 1], \,[F \, 2], \,[F \, 3]\).

Another special case, which includes the preceding one, occurs when the \( n-1 \) dimensional integralgeometric measure of the boundary of \( A \) is finite. Here the validity of (1) may be derived from known characterizations \([F \, 4], [K]\) of distributions whose partial derivatives are measures.

Finally consider a region \( A \) whose boundary \( B \) is a finitely triangulable \( n-1 \) dimensional manifold. In case \( n = 2 \) the condition (1) holds if and only if the simple closed curve \( B \) has finite length. For \( n \geq 3 \) the situation is much more complicated. If the inclusion map \( b \) of \( B \) into \( E_n \) has finite \( n-1 \) dimensional Lebesgue area and if \( L_n(B) \)
=0, then (1) holds. It would be interesting to know whether (1) implies that the Lebesgue area of \( b \) is finite.

References


DG 2. ———, *Nuovi teoremi relativi alle misure \( r-1 \) dimensionali in uno spazio ad \( r \) dimensioni*, Ricerche di Matematica vol. 4 (1955) p. 95.


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