ON THE GROUP OF AFFINITIES OF LOCALLY AFFINE SPACES

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Let $M$ be a compact manifold with a given complete flat affine connection (i.e., an affine connection with curvature and torsion zero). Then we may represent the fundamental group $\Gamma$ of $M$ by affine transformations of the real affine space $\mathbb{R}^n$, in such a way that the orbit space of $\mathbb{R}^n$ by $\Gamma$ is homeomorphic to $M$. We will denote the full group of affine transformations of $\mathbb{R}^n$ by $A(n)$ and the orbit space of $\mathbb{R}^n$ under $\Gamma$ by $\mathbb{R}^n/\Gamma$. We represent the elements of $A(n)$ as matrices of the form

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$$

where $A$ is the (nonsingular) linear transformation part and $v$ is the translational part. Let $G$ be the group of all affinities of $M$, i.e., the group of all homeomorphisms of $M$ onto itself which preserve the given affine structure on $M$. Nomizu proved in [3] that $G$ is a Lie group. Let $G_1$ denote the identity component of $G$. It is the purpose of this note to prove that $G_1$ is a nilpotent Lie group.

Now it is well known that any map of $M$ into itself can be lifted to a map of $\mathbb{R}^n$ into itself, uniquely up to covering transformations, i.e., up to elements of $\Gamma$. The maps in $G_1$ lift to affine transformations of $\mathbb{R}^n$. It is clear that $G^*$, the identity component of the subgroup of $A(n)$ so obtained, projects back onto $G_1$ as a covering group. Further, since $g^*Tg^{-1}=\Gamma$, for all $g^*\in G^*$ and since $G^*$ is connected and $\Gamma$ discrete, it follows easily that $G^*$ and $\Gamma$ commute elementwise.

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Lemma 1. There exist $\gamma_i \in \Gamma$, $i = 1, \cdots, n$, whose translational components are linearly independent.

Proof. Assume the lemma is false and all translational components lie in a linear subspace $V \subset \mathbb{R}^n$. Then $V$ must be invariant by $\Gamma$ that is, we have $\gamma(V) \subset V$ for all $\gamma \in \Gamma$. In $\mathbb{R}^n$ choose a compact fundamental domain $D$ for $\mathbb{R}^n/\Gamma$. Then $V \cap D$ is compact and a fundamental domain for $\Gamma$ restricted to $V$. Hence $V/\Gamma$ must be a compact manifold of dimension less than $n$ with fundamental group $\Gamma$. Using the theorem of Eilenberg-MacLane on groups operating on acyclic spaces [2], we see that the $n$ dimensional cohomology group of the group $\Gamma$ with coefficients integers modulo 2, must be zero. But this contradicts the fact that $\Gamma$ is also the fundamental group of an $n$ dimensional manifold with $\mathbb{R}^n$ as universal covering space.

Lemma 2. Let $g^* \in G^*$ be such that $g^*(x_0) = x_0$ for some $x_0 \in \mathbb{R}^n$. Then $g^*$ is the identity element of $G^*$.

Proof. Let $g^*(x_0) = x_0$. Choose $x_0$ as the origin of the coordinate system. Now $g^*g(x_0) = y(x_0)$. Hence $g^*$ leaves the images of $x_0$ under $\Gamma$ point-wise fixed. Since the points $\gamma(x_0), \gamma \in \Gamma$ span $\mathbb{R}^n$ by Lemma 1, $g^*$ is the identity element of $G^*$.

Theorem. Let $M$ be a complete compact flat affine space. Let $G$ be its group of affinities. Then the identity component of $G$ is a nilpotent Lie group.

This is equivalent to proving that $G^*$ is a nilpotent Lie group. Now $G^* \subset A(n)$. Let $\mathbb{C}^n$ denote the $n$-dimensional affine space over the complex field; the corresponding group of affine transformations will be denoted by $A(n, \mathbb{C})$. Then $A(n)$ may be considered as a subgroup of $A(n, \mathbb{C})$. Since $g^* \in G^*$ operates without fixed points in $\mathbb{R}^n$, it will do so also in $\mathbb{C}^n$. Further $\gamma g^* = g^*\gamma$.

Let $\exp(Mt)$ be a one parameter subgroup of $G^*$ with infinitesimal generator $M$. The matrix $M$ can be assumed in normal form

\[
\begin{pmatrix}
M_1 & 0 & v_1 \\
.M_2 & . & . \\
0 & . & . \\
0 & . & M_k \\
0 & v_k 
\end{pmatrix}
\]

where each $M_i$ is triangular with all eigenvalues equal.

Lemma 3. All eigenvalues of $M$ are 0; i.e., $M$ is nilpotent.
Proof. By changing the origin, if necessary, we can assume that the translational part $v_i$ is 0 for every $M_i$ with eigenvalue different from 0. At least one eigenvalue must be 0, otherwise the elements $\exp(Mt)$ would have the origin as fixed point; this would contradict Lemma 2. Because $M$ has the above special form, it is easy to see that the coordinates of the points on the orbit of the origin under $\exp(Mt)$ are given by polynomials in $t$. Suppose that there is an $M_i$ with eigenvalue not zero. Let $y$ be any point with a nonzero coordinate corresponding to the element in position $(1, 1)$ of $M_i$. But then on the orbit of $y$ under $\exp(Mt)$ this coordinate has the form $C \cdot \exp(\lambda t)$, with $C \neq 0$. Applying Lemma 1, one sees that there exists a $\gamma \in \Gamma$ such that for the orbit of $\gamma(0)$ at least one coordinate is of exponential form. But because $g^*(\gamma(0)) = \gamma(g^*(0))$ and the remark on the orbit of 0 above, the coordinates on the orbit of $\gamma(0)$ must be polynomials. This proves Lemma 3.

Our theorem is now an immediate consequence of the well-known fact that a linear Lie algebra, all of whose elements are nilpotent, is nilpotent [4].

Bibliography


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