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ON THE GROUP OF AFFINITIES OF LOCALLY AFFINE SPACES

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Let M be a compact manifold with a given complete flat affine connection (i.e., an affine connection with curvature and torsion zero). Then we may represent the fundamental group Γ of M by affine transformations of the real affine space R^n, in such a way that the orbit space of R^n by Γ is homeomorphic to M. We will denote the full group of affine transformations of R^n by $A(n)$ and the orbit space of R^n under Γ by R^n/Γ. We represent the elements of $A(n)$ as matrices of the form

$$\left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right)$$

where A is the (nonsingular) linear transformation part and v is the translational part. Let G be the group of all affinities of M, i.e., the group of all homeomorphisms of M onto itself which preserve the given affine structure on M. Nomizu proved in [3] that G is a Lie group. Let G_1 denote the identity component of G. It is the purpose of this note to prove that G_1 is a nilpotent Lie group.

Now it is well known that any map of M into itself can be lifted to a map of R^n into itself, uniquely up to covering transformations, i.e., up to elements of Γ. The maps in G_1 lift to affine transformations of R^n. It is clear that G^*, the identity component of the subgroup of $A(n)$ so obtained, projects back onto G_1 as a covering group. Further, since $g^*\Gamma g^{*-1}=\Gamma$, for all $g^*\in G^*$ and since G^* is connected and Γ discrete, it follows easily that G^* and Γ commute elementwise.

Received by the Editors January 4, 1957.

[1] This paper was written while the author was a National Science Foundation Post Doctoral Fellow.

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LEMMA 1. *There exist $\gamma_i \in \Gamma$, $i=1, \cdots, n$, whose translational components are linearly independent.*

PROOF. Assume the lemma is false and all translational components lie in a linear subspace $V \subset R^n$. Then V must be invariant by Γ that is, we have $\gamma(V) \subset V$ for all $\gamma \in \Gamma$. In R^n choose a compact fundamental domain D for R^n/Γ. Then $V \cap D$ is compact and a fundamental domain for Γ restricted to V. Hence V/Γ must be a compact manifold of dimension less than n with fundamental group Γ. Using the theorem of Eilenberg-MacLane on groups operating on acyclic spaces [2], we see that the n dimensional cohomology group of the group Γ with coefficients integers modulo 2, must be zero. But this contradicts the fact that Γ is also the fundamental group of an n dimensional manifold with R^n as universal covering space.

LEMMA 2. *Let $g^* \in G^*$ be such that $g^*(x_0) = x_0$ for some $x_0 \in R^n$. Then g^* is the identity element of G^*.*

PROOF. Let $g^*(x_0) = x_0$. Choose x_0 as the origin of the coordinate system. Now $g^*\gamma(x_0) = \gamma g^*(x_0) = \gamma(x_0)$. Hence g^* leaves the images of x_0 under Γ point-wise fixed. Since the points $\gamma(x_0)$, $\gamma \in \Gamma$ span R_n by Lemma 1, g^* is the identity element of G^*.

THEOREM. *Let M be a complete compact flat affine space. Let G be its group of affinities. Then the identity component of G is a nilpotent Lie group.*

This is equivalent to proving that G^* is a nilpotent Lie group. Now $G^* \subset A(n)$. Let C^n denote the n-dimensional affine space over the complex field; the corresponding group of affine transformations will be denoted by $A(n, C)$. Then $A(n)$ may be considered as a subgroup of $A(n, C)$. Since $g^* \in G^*$ operates without fixed points in R^n, it will do so also in C^n. Further $\gamma g^* = g^* \gamma$.

Let exp (Mt) be a one parameter subgroup of G^* with infinitesimal generator M. The matrix M can be assumed in normal form

$$\left(\begin{array}{cccc|c} M_1 & & & 0 & v_1 \\ & M_2 & & & v_2 \\ & & \ddots & & \vdots \\ 0 & & & M_k & v_k \\ \hline & & 0 & & 0 \end{array}\right)$$

where each M_i is triangular with all eigenvalues equal.

LEMMA 3. *All eigenvalues of M are* 0; *i.e., M is nilpotent.*

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PROOF. By changing the origin, if necessary, we can assume that the translational part v_i is 0 for every M_i with eigenvalue different from 0. At least one eigenvalue must be 0, otherwise the elements $\exp(Mt)$ would have the origin as fixed point; this would contradict Lemma 2. Because M has the above special form, it is easy to see that the coordinates of the points on the orbit of the origin under $\exp(Mt)$ are given by polynomials in t. Suppose that there is an M_i with eigenvalue not zero. Let y be any point with a nonzero coordinate corresponding to the element in position (1, 1) of M_i. But then on the orbit of y under $\exp(Mt)$ this coordinate has the form $C\cdot\exp(\lambda_i t)$, with $C\neq 0$. Applying Lemma 1, one sees that there exists a $\gamma\in\Gamma$ such that for the orbit of $\gamma(0)$ at least one coordinate is of exponential form. But because $g^*(\gamma(0))=\gamma(g^*(0))$ and the remark on the orbit of 0 above, the coordinates on the orbit of $\gamma(0)$ must be polynomials. This proves Lemma 3.

Our theorem is now an immediate consequence of the well-known fact that a linear Lie algebra, all of whose elements are nilpotent, is nilpotent [**4**].

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