DENSITY THEOREMS FOR OUTER MEASURES IN n-SPACE[1]

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1. **Introduction.** In Euclidean n-space R^n let $L_n(E)$ be the Lebesgue n-dimensional outer measure of the set $E \subset R^n$. The closed sphere with center at a point $x \in R^n$ and radius $r > 0$ will be denoted by $C(x, r)$. If E is an L_n-measurable set in R^n then it is well known that

$$(1) \qquad \lim_{r \to 0} \frac{L_n[E \cap C(x, r)]}{r^n} = 0 \text{ for } L_n\text{-a.e. } x \in \mathcal{C}E,$$

where $\mathcal{C}E$ is the complement of the set E.

A well known generalization of the above result is that if E is an L_n-measurable set in R^n then

$$(2) \qquad \lim_{r \to 0} \frac{\Gamma[E \cap C(x, r)]}{r^n} = 0 \text{ or } \infty \text{ for } L_n\text{-a.e. } x \in \mathcal{C}E,$$

if Γ is a Carathéodory outer measure in R^n, i.e., $\Gamma(E)$ is defined for all sets $E \subset R^n$ such that $0 \leqq \Gamma(E) \leqq \infty$ and satisfies the following conditions.

(i) $\Gamma(\emptyset) = 0$ where $\emptyset$ is the empty set.

(ii) $\Gamma(E_1) \leqq \Gamma(E_2)$ if $E_1 \subset E_2$.

(iii) If $E_1, E_2, \cdots$, is a sequence of sets then $\Gamma(\bigcup_i E_i) \leqq \sum_i \Gamma(E_i)$.

(iv) If the distance between two sets E_1, E_2 is positive then $\Gamma(E_1 \cup E_2) = \Gamma(E_1) + \Gamma(E_2)$.

It is the purpose of this note to generalize the result in (2) as follows. A set function Ψ defined for all sets $E \subset R^n$ such that $0 \leqq \Psi(E) \leqq \infty$ and satisfying the above conditions (i), (ii) and (iii) will be called simply an outer measure. In §2 we show that if Ψ is an outer measure in R^n and E is an L_n-measurable set in R^n then

$$(3) \qquad \limsup_{r \to 0} \frac{\Psi[E \cap C(x, r)]}{r^n} = 0 \text{ or } \infty \text{ for } L_n\text{-a.e. } x \in \mathcal{C}E.$$

The fact that the limit superior cannot be replaced by the limit in (3) is shown by an example in §4. In §3 we show that the result in (3) can be used to derive the result in (2).

While the results stated here are adequate for application to some

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recent work of the writers in surface area theory, some further generalizations will be discussed in a later more comprehensive paper.

2. **Outer measures.** For an outer measure Ψ in R^n we set

$$(1) \qquad D^*(\Psi) = \left\{ x \mid x \in R^n, \limsup_{r\to 0} \frac{\Psi[C(x, r)]}{r^n} < \infty \right\}.$$

LEMMA 1. *If Ψ is an outer measure in R^n then for every pair of real numbers a, b satisfying $0<a<\infty$, $0<b<\infty$ the set*

$$D(a, b) = \{x \mid x \in R^n, \Psi[C(x, r)] \leqq br^n \text{ for } 0 < r < a\}$$

is a closed set and

$$(2) \qquad D^*(\Psi) = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} D(1/i, j).$$

PROOF. Let $x_i \in D(a, b)$, $i=1, 2, \cdots$, be a sequence of points such that $\lim_{i\to\infty} x_i = x_0$. For $0<r<a$ let r' be a number satisfying $0<r<r'<a$. For i sufficiently large $C(x_0, r) \subset C(x_i, r')$ and for this i

$$(3) \qquad \Psi[C(x_0, r)] \leqq \Psi[C(x_i, r')] \leqq b(r')^n.$$

Since (3) holds for all r' such that $0<r<r'<a$ it follows that $\Psi[C(x_0, r)] \leqq br^n$ for $0<r<a$. Thus $x_0 \in D(a, b)$ and $D(a, b)$ is a closed set. The relation (2) is easily verified.

LEMMA 2. *Let Ψ be an outer measure in R^n and let F be a closed set in R^n for which there exist constants K, δ satisfying $0<K<\infty$, $0<\delta<\infty$ such that*

$$(4) \qquad \Psi[C(x, r)] \leqq Kr^n \text{ for } 0 < r < \delta \text{ and } C(x, r) \cap F \neq \emptyset.$$

Then

$$(5) \qquad \lim_{r\to 0} \frac{\Psi[\mathcal{C}F \cap C(x, r)]}{r^n} = 0 \text{ for } L_n\text{-a.e. } x \in F.$$

PROOF. Set $\mathcal{O} = \mathcal{C}F$. Take a point $x \in F$ and a number r such that $0<r<\delta/5$. For $y \in \mathcal{O} \cap C(x, r)$ let r_y be $1/2$ the distance from y to F. Then

$$(6) \qquad C(y, r_y) \subset \mathcal{O} \cap C(x, 2r),$$

$$(7) \qquad C(y, 5r_y) \cap F \neq \emptyset, \qquad 5r_y < 5r < \delta,$$

$$(8) \qquad \mathcal{O} \cap C(x, r) \subset \bigcup C(y, r_y) \text{ for } y \in \mathcal{O} \cap C(x, r).$$

From (8) it follows by a result of A. P. Morse [1, Theorem 3.5] (numbers in square brackets refer to the bibliography at the end of this paper) that there is a countable disjoint sequence of these spheres $C(y_1, r_1)$, $C(y_2, r_2)$, $\cdots$, $r_i = r_{y_i}$, such that

$$\mathcal{O} \cap C(x, r) \subset \bigcup_i C(y_i, 5r_i). \tag{9}$$

From (9) it follows that

$$\Psi[\mathcal{O} \cap C(x, r)] \leqq \sum_i \Psi[C(y_i, 5r_i)]. \tag{10}$$

From (7) and (4) it follows that

$$\Psi[C(y_i, 5r_i)] \leqq 5^n K r_i^n \leqq \frac{5^n K L_n[C(y_i, r_i)]}{\alpha(n)}, \tag{11}$$

where $\alpha(n)$ is the L_n-measure of the unit n-sphere in R^n. Since the spheres $C(y_1, r_1)$, $C(y_2, r_2)$, $\cdots$, are disjoint, from (6)

$$\sum_i L_n[C(y_i, r_i)] \leqq L_n[\mathcal{O} \cap C(x, 2r)]. \tag{12}$$

From (10), (11) and (12) we have

$$\frac{\Psi[\mathcal{O} \cap C(x, r)]}{r^n} \leqq \frac{10^n K}{\alpha(n)} \frac{L_n[\mathcal{O} \cap C(x, 2r)]}{(2r)^n}. \tag{13}$$

Since the limit for $r \to 0$ of the right side of (13) is 0 for L_n-a.e. $x \in F$ the same holds for the left side of (13). Thus (5) holds.

LEMMA 3. *Let Ψ be an outer measure in R^n and let $E \subset D^*(\Psi)$ be an L_n-measurable set. Then*

$$\lim_{r \to 0} \frac{\Psi[\mathcal{C}E \cap C(x, r)]}{r^n} = 0 \textit{ for } L_n\textit{-a.e. } x \in E. \tag{14}$$

PROOF. Since E is L_n-measurable it is the union of a countable number of closed sets and a set of L_n-measure zero. Hence, to prove (14) it is sufficient to assume that E is a closed set. By Lemma 1 for $F_{ij} = E \cap D(1/i, j)$, $i, j = 1, 2, \cdots$, F_{ij} is a closed set and

$$E = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} F_{ij}. \tag{15}$$

Set $\delta = 1/2i$, $K = 2^n j$. If $0 < r < \delta$ and $C(y, r) \cap F_{ij} \neq \emptyset$ then for $x \in C(y, r) \cap F_{ij}$ we have $C(y, r) \subset C(x, 2r)$. Thus

$$\Psi[C(y, r)] \leqq \Psi[C(x, 2r)] \leqq 2^n j r^n = K r^n.$$

By Lemma 2 it follows that

$$\lim_{r\to 0} \frac{\Psi[\complement F_{ij} \cap C(x, r)]}{r^n} = 0 \text{ for } L_n\text{-a.e. } x \in F_{ij}.$$

Since $\Psi[\complement E \cap C(x, r)] \leqq \Psi[\complement F_{ij} \cap C(x, r)]$ we have

$$\lim_{r\to 0} \frac{\Psi[\complement E \cap C(x, r)]}{r^n} = 0 \text{ for } L_n\text{-a.e. } x \in F_{ij}. \tag{16}$$

Since the right side of (15) is a countable union of sets, (14) follows from (15) and (16).

THEOREM 1. *Let Ψ be an outer measure in R^n and let E_0 be an L_n-measurable set in R^n. Then*

$$\limsup_{r\to 0} \frac{\Psi[E_0 \cap C(x, r)]}{r^n} = 0 \textit{ or } \infty \textit{ for } L_n\text{-a.e. } x \in \complement E_0. \tag{17}$$

PROOF. Set $\Psi^*(E) = \Psi(E \cap E_0)$. It is easily verified that Ψ^* is an outer measure. Then

$$\limsup_{r\to 0} \frac{\Psi^*[C(x, r)]}{r^n} = \infty \text{ for } x \in \complement E_0 - D^*(\Psi^*) \tag{18}$$

and, since $\complement E_0 \cap D^*(\Psi^*)$ is an L_n-measurable set, by Lemma 3

$$\lim_{r\to 0} \frac{\Psi^*[E_0 \cap C(x, r)]}{r^n} = 0 \text{ for } L_n\text{-a.e. } x \in D^*(\Psi^*) \cap \complement E_0. \tag{19}$$

Since $\Psi^*[C(x, r)] = \Psi^*[E_0 \cap C(x, r)] = \Psi[E_0 \cap C(x, r)]$, (17) follows from (18) and (19).

3. Carathéodory outer measures.

LEMMA 4. *Let Γ be a Carathéodory outer measure in R^n and let $E \subset R^n$ be such that $\Gamma(E) < \infty$. Then for*

$$G = \left\{ x \,\middle|\, x \in R^n, \limsup_{r\to 0} \frac{\Gamma[E \cap C(x, r)]}{r^n} = \infty \right\}$$

we have $L_n(G) = 0$.

PROOF. For a positive integer k let $\mathfrak{F}$ be the family of closed spheres $C(x, r)$ such that $\Gamma[E \cap C(x, r)] > kr^n$. Then $\mathfrak{F}$ covers G in the sense of Vitali. Thus there is a disjoint countable sequence of them $C(x_i, r_i)$, $i = 1, 2, \cdots$, such that

$$L_n\left[G - \bigcup_i C(x_i, r_i)\right] = 0.$$

Hence

$$(1)\quad L_n(G) \leqq \sum_i L_n[C(x_i, r_i)] = \alpha(n) \sum_i r_i^n \leqq \frac{\alpha(n)}{k} \sum_i \Gamma[E \cap C(x_i, r_i)]$$

where $\alpha(n)$ is the L_n-measure of the unit n-sphere in R^n. Since the $C(x_1, r_1)$, $C(x_2, r_2)$, $\cdots$, are disjoint Γ-measurable sets, by Saks [2, Theorem 4.6]

$$(2)\qquad \sum_i \Gamma[E \cap C(x_i, r_i)] = \Gamma\left[E \cap \left(\bigcup_i C(x_i, r_i)\right)\right] \leqq \Gamma(E).$$

From (1) and (2) we have

$$(3)\qquad L_n(G) \leqq \frac{\alpha(n)}{k} \Gamma(E).$$

Since $\Gamma(E) < \infty$ and (3) holds for every positive integer k, it follows that $L_n(G) = 0$.

THEOREM 2. *Let Γ be a Carathéodory outer measure in R^n and let E be an L_n-measurable set in R^n such that $\Gamma(E) < \infty$. Then*

$$(4)\qquad \lim_{r \to 0} \frac{\Gamma[E \cap C(x, r)]}{r^n} = 0 \text{ for } L_n\text{-a.e. } x \in \complement E.$$

PROOF. By Theorem 1

$$(5)\qquad \limsup_{r \to 0} \frac{\Gamma[E \cap C(x, r)]}{r^n} = 0 \text{ or } \infty \text{ for } L_n\text{-a.e. } x \in \complement E.$$

By Lemma 4, since $\Gamma(E) < \infty$,

$$(6)\qquad \limsup_{r \to 0} \frac{\Gamma[E \cap C(x, r)]}{r^n} < \infty \text{ for } L_n\text{-a.e. } x \in \complement E.$$

(4) then follows from (5) and (6).

THEOREM 3. *Let Γ be a Carathéodory outer measure in R^n, let E be an L_n-measurable set in R^n and let*

$$H = \{x \mid x \in R^n, \Gamma[E \cap C(x, r)] = \infty \text{ for all } r > 0\}.$$

Then

$$(7)\qquad \lim_{r\to 0}\frac{\Gamma[E\cap C(x,r)]}{r^n}=0 \textit{ for } L_n\text{-a.e. } x\in \complement H\cap \complement E$$

and

$$(8)\qquad \lim_{r\to 0}\frac{\Gamma[E\cap C(x,r)]}{r^n}=\infty \textit{ for } x\in H\cap \complement E.$$

PROOF. (8) follows immediately from the definition of H. It follows readily that H is a closed set. Let $K(x, r)$ be the open sphere with center at x and radius $r>0$. For each $x\in\complement H$ there is an r_x such that $K(x, r_x)\subset\complement H$ and $\Gamma[E\cap K(x, r_x)]<\infty$. Thus there is a countable number of open spheres $K(x_1, r_1)$, $K(x_2, r_2)$, $\cdots$, such that

$$(9)\qquad \complement H=\bigcup_i K(x_i, r_i),\ \Gamma[E\cap K(x_i, r_i)]<\infty,\quad i=1, 2, \cdots.$$

Since $\Gamma[E\cap C(x, r)]=\Gamma[E\cap C(x, r)\cap K(x_i, r_i)]$ for $x\in K(x_i, r_i)$ and r sufficiently small and $\Gamma[E\cap K(x_i, r_i)]<\infty$ it follows from Theorem 2 that for $i=1, 2, \cdots$,

$$(10)\qquad \lim_{r\to 0}\frac{\Gamma[E\cap C(x,r)]}{r^n}=0 \text{ for } L_n\text{-a.e. } x\in K(x_i, r_i)\cap \complement E.$$

(7) now follows from (9) and (10).

It should be noted that if Γ is locally finite then $H=\emptyset$ and the result in the above Theorem 3 is entirely analogous to the result for Lebesgue measure L_n noted in (1) of the introduction.

4. **An example.** We now give an example to show that the limit superior in Theorem 1 cannot be replaced by the limit. On the closed interval $I: 0\leqq x\leqq 1$ of R^1 let F be a nowhere dense closed set of positive L_1-measure. For two sets E_1, E_2 in I

$$d(E_1, E_2) = \text{gr.l.b. } |x_1-x_2| \quad \text{for } x_1\in E_1,\ x_2\in E_2.$$

Since F is nowhere dense, for each positive integer k there is a set F_k consisting of a finite number of points in $I-F$ such that

$$d(x, F_k) < 1/k \text{ for } x\in F.$$

Let $k_1<k_2<\cdots$ be a sequence of positive integers satisfying

$$(1)\qquad 1/k_i < d(F, F_{k_{i-1}})/2 < 1/2k_{i-1}.$$

We first show that for $x\in F$

$$(2)\qquad C(x, 1/k_i)\cap F_{k_j}\begin{cases} =\emptyset & \text{for } j<i,\\ \neq\emptyset & \text{for } j\geqq i.\end{cases}$$

For $j<i$, $d(x, F_{k_j}) \geqq d(F, F_{k_j}) \geqq 2/k_i$ and $C(x, 1/k_i) \cap F_{k_j} = \emptyset$. For $j \geqq i$, $d(x, F_{k_j}) < 1/k_j \leqq 1/k_i$ and $C(x, 1/k_i) \cap F_{k_j} \neq \emptyset$.

For $E \subset R^1$ set

$$\Psi_j(E) = \begin{cases} 0 & \text{if } E \cap F_{k_j} = \emptyset, \\ 1/k_j & \text{if } E \cap F_{k_j} \neq \emptyset \end{cases}$$

and

$$\Psi(E) = \sum_{j=1}^{\infty} \Psi_j(E).$$

It is easily verified that $\Psi(E)$ is an outer measure. Set $E_0 = \bigcup_{j=1}^{\infty} F_{k_j}$. By (2) and (1) for $x \in F$ and $r_i = 1/k_i$,

$$1 = \frac{r_i}{r_i} \leqq \frac{\Psi[E_0 \cap C(x, r_i)]}{r_i} \leqq \frac{2r_i}{r_i} = 2.$$

Since $r_i \to 0$ for $i \to \infty$,

$$\liminf_{r \to 0} \frac{\Psi[E_0 \cap C(x, r)]}{r} \leqq 2 \text{ for } x \in F, \tag{3}$$

$$\limsup_{r \to 0} \frac{\Psi[E_0 \cap C(x, r)]}{r} \geqq 1 \text{ for } x \in F. \tag{4}$$

By (4) and Theorem 1, since $F \subset \mathcal{C}E_0$,

$$\limsup_{r \to 0} \frac{\Psi[E_0 \cap C(x, r)]}{r} = \infty \text{ for } L_1\text{-a.e. } x \in F. \tag{5}$$

By (3) and (5) the limit does not exist for L_1-a.e. $x \in F$, $F \subset \mathcal{C}E_0$ and $L_1(F) > 0$.

Bibliography

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