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A MULTIPLICATION THEOREM FOR POSITIVE REAL FUNCTIONS[1]

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Modern network synthesis relies heavily on the use of Positive Real Functions. A function $Z(s)$ of the complex frequency s is said to be a P.R.F. (short for Positive Real Function) when it satisfies the following three conditions.

(a) $Z(s)$ is analytic and single valued for $\operatorname{Re} s \geqq 0$ except possibly for poles on the imaginary axis,

(b) $Z(s)$ is real for real s,

(c) $\operatorname{Re} Z(s) \geqq 0$ for $\operatorname{Re} s \geqq 0$.

When in addition $\operatorname{Re} Z(iy) = 0$, the function $Z(s)$ will be called an I.P.R.F.

The object of this note is to contribute two theorems clarifying the "algebra" of Positive Real Functions, and providing a key for their generation. These theorems employ a simple normalization procedure (operation N) after which the product of any number of normalized P.R.F.'s is a normalized P.R.F. Since obviously the sum of several P.R.F.'s is a P.R.F. then one will obtain a rather simple algebra for these functions. One may symbolically write:

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$$\sum \text{P.R.F.} = \text{P.R.F.}, \tag{1}$$

$$N^{-1}\left\{\prod [N(\text{P.R.F.})]\right\} = \text{P.R.F.} \tag{2}$$

The symbols $\sum$ and $\prod$ stand for sum and product respectively. N stands for the operation which will be described below. The consequences of these basic theorems are broad. Corollaries of these theorems include some of the results previously obtained on the behavior of electrical networks in [**1**; **2**; **3**; **4**].

THEOREM 1. *Let $z_1, z_2, \cdots, z_k, \cdots, z_n$ be P.R.F. Then:*

$$Z = \frac{\prod_1^n (1 + z_k) - \prod_1^n (1 - z_k)}{\prod_1^n (1 + z_k) + \prod_1^n (1 - z_k)} \tag{3}$$

is a P.R.F

PROOF. For $\operatorname{Re} s \geqq 0$ we have by definition $\operatorname{Re} z_k \geqq 0$. The transformation:

$$(\text{Operation } N) \quad w_k = \frac{1 - z_k}{1 + z_k} = N(z_k) \tag{4}$$

maps the right half of the z_k-plane onto the unit disc of the w_k-plane. Now consider the function:

$$W = \prod_1^n w_k. \tag{5}$$

For a point of the s-plane with $\operatorname{Re} s \geqq 0$ one obtains a corresponding image point in the W-plane. The latter point will necessarily lie within or on the unit circle. This because the product of several numbers all in or on the unit circle cannot be outside of the unit circle. The transformation:

$$Z = \frac{1 - W}{1 + W} \tag{6}$$

maps the unit circle of the W-plane onto the right half of the Z-plane. Thus $Z(s)$ satisfies conditions (a), (b) and (c) and so is a P.R.F.

$$Z(s) = \frac{1 - \prod_1^n w_k}{1 + \prod_1^n w_k} = \frac{\prod_1^n (1 + z_k) - \prod_1^n (1 - z_k)}{\prod_1^n (1 + z_k) + \prod_1^n (1 - z_k)}. \tag{7}$$

COROLLARY 1. *Let z_k, $k=1, 2, \cdots, n$, be I.P.R.F.'s then $Z(s)$ as described in* (3) *will also be an I.P.R.F.*

PROOF. z_k will map the axis of imaginaries of the s-plane onto the axis of imaginaries of the z_k-plane and the unit circle of the w_k-plane. This will lead in turn to the unit circle of the W-plane for the corresponding map of $\operatorname{Re} s=0$ by the W function. Finally the unit circle of the W-plane will be mapped onto the axis of imaginaries of the Z-plane.

An Inverse Theorem. We have proved that the product of normalized P.R.F.'s in the above sense is a normalized P.R.F. The inverse of the operation (3) has not yet been discussed. For simplicity consider the case of two P.R.F.'s z_1 and z_2. On the basis of the preceding theorem

$$Z = \frac{(1+z_1)(1+z_2)-(1-z_1)(1-z_2)}{(1+z_1)(1+z_2)+(1-z_1)(1+z_2)} = \frac{z_1+z_2}{1+z_1 z_2} \tag{8}$$

is a P.R.F. Now the question is to see whether for a given pair of P.R.F.'s, say Z and z_2, one can always find a P.R.F. z_1 such that (9) is satisfied. Let

$$z_1 = \frac{Z - z_2}{1 - z_2 Z} \cdot \tag{9}$$

Apply transformation (4) to z_1:

$$\frac{1-z_1}{1+z_1} = \frac{1 - \dfrac{Z-z_2}{1-z_2 Z}}{1 + \dfrac{Z-z_2}{1-z_2 Z}} = \frac{(1-Z)(1+z_2)}{(1+Z)(1-z_2)}, \tag{10}$$

$$w_1 = W \cdot w_2^{-1}, \tag{11}$$

W and w_2 are by definition confined to the unit circle. w_1 will not necessarily be generally confined to the unit circle. The condition under which this is possible is given by:

$$\left| W \cdot w_2^{-1} \right| \leqq 1 \qquad \text{for } \operatorname{Re} s \geqq 0. \tag{12}$$

Note that if z_2 is an I.P.R.F. then

$$\left| w_2 \right| = \left| w_2^{-1} \right| = 1 \qquad \text{for } \operatorname{Re} s = 0. \tag{13}$$

If all poles and zeros of w_2 coincide respectively with some of the zeros and poles of w, then w_1 will be less complex than w. This will in turn lead to a less complex P.R.F. when the condition (12) is satisfied.

COROLLARY 2. *Let* z_2 *be an I.P.R.F.,* Z *a general P.R.F. then* z_1 *also is a P.R.F., provided* (12) *is satisfied and* $Zz_2 \neq 1$,

$$z_1 = \frac{Z - z_2}{1 - z_2 Z}. \tag{14}$$

THEOREM 2. *Let* a_k, $k=0, 1, 2, \cdots, n$ *be real numbers such that* $|a_k| \leqq 1$, *let* Z *be any P.R.F., and let*

$$P(NZ) = \frac{1}{n+1}\left[a_0(NZ)^n + a_1(NZ)^{n-1} + \cdots + a_{n-1}(NZ) + a_n\right]. \tag{15}$$

Then the normalized polynomial $N[P(NZ)]$ *will also be a P.R.F.*

This theorem may be readily proved, and the proof will be omitted here.

In the light of the preceding theorems one can easily establish a variety of corollaries pertinent to the behavior of P.R.F.

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