THE MEASURE OF THE SET OF ADMISSIBLE LATTICES

WOLFGANG SCHMIDT

Introduction. Let S be a Borel set in n-dimensional space which does not contain the origin 0. We assume that there is no X so that both $X \in S$ and $-X \in S$. We say a point lattice Λ is S-admissible, if there is no lattice point of Λ in S. We denote by $A(S)$ the set of S-admissible lattices and by $V = V(S)$ the measure of S.

The main result of this paper is

THEOREM 4. *If*

$$V \leqq n - 1 \quad \text{and} \quad n \geqq 13, \tag{1}$$

then

$$m(A(S)) = \int_{\Omega\Lambda_0 \in A(S);\, \Omega \in F} d\mu(\Omega) = e^{-V}(1 - R), \tag{2}$$

where

$$|R| < 6(3/4)^{n/2} e^{4V} + V^{n-1} n^{-n+1} e^{V+n}. \tag{3}$$

Here Ω denotes a linear transformation of determinant 1, F is a fundamental region with respect to the subgroup of unimodular transformations of determinant 1, and $\mu(\Omega)$ is the invariant measure on the space of linear transformations with determinant 1, defined by C. L. Siegel [5], normalized so that

$$\int_F d\mu(\Omega) = 1. \tag{4}$$

Λ_0 denotes the lattice of points with integral coordinates.

Theorem 4 will be used to prove Theorem 5 which is an improvement of the *Minkowski-Hlawka* Theorem. We also prove two existence theorems which are in a certain sense converses of the *Minkowski-Hlawka* Theorem (Theorem 6 and Theorem 7).

The author is very indebted to the referee who pointed out some errors of the originally submitted paper and made useful suggestions. The originally stated Theorem 1 was wrong.

Presented to the Society, June 15, 1957; received by the editors November 23, 1956 and, in revised form, June 3, 1957 and October 31, 1957.

1. We define the lattice function

$$\alpha(\Lambda) = \begin{cases} 1, & \text{for } \Lambda \in A(S), \\ 0, & \text{for } \Lambda \notin A(S), \end{cases}$$

and $\rho(\Lambda)$ to be the number of lattice points of Λ in S. The usual bound for $\alpha(\Lambda)$, used for the proof of the *Minkowski-Hlawka* Theorem, is

$$\alpha(\Lambda) \geqq 1 - \rho(\Lambda). \tag{5}$$

In §1 we shall replace (5) by a better bound.

We define for $0 \leqq j \leqq k \leqq n$, $k > 0$,

$$\overset{j}{\rho}_k(\Lambda)$$

to be the number of k-tuples $(X_1, \cdots, X_k)$ of different lattice points X_i of Λ with $X_1 \in S, \cdots, X_k \in S$ and $\dim (X_1, \cdots, X_k) = j$. (Here the order is immaterial, that is, we count k points of a k-tuple $(X_1, \cdots, X_k)$ only once and not $k!$ times.)

We further define $\tau_k(\Lambda)$ and $\pi_k(\Lambda)$ by

$$\tau_k(\Lambda) = \begin{cases} \overset{k}{\rho}_k(\Lambda), \text{ if } k \text{ is even}, \\ \overset{k}{\rho}_k(\Lambda) + \overset{k-1}{\rho}_k(\Lambda), \text{ if } k \text{ is odd}, \end{cases}$$

and

$$\pi_k(\Lambda) = \begin{cases} \rho_k(\Lambda), \text{ if } k \text{ is odd}, \\ \overset{k}{\rho}_k(\Lambda) + \overset{k-1}{\rho}_k(\Lambda), \text{ if } k \text{ is even}. \end{cases}$$

Since $0 \notin S$, $\tau_1(\Lambda) = \rho_1^1(\Lambda) + \rho_1^0(\Lambda) = \rho_1^1(\Lambda) = \rho(\Lambda)$.

The purpose of this section is to prove

THEOREM 1.

$$1 + \sum_{k=1}^{g} (-1)^k \pi_k(\Lambda) \geqq \alpha(\Lambda) \geqq 1 + \sum_{k=1}^{h} (-1)^k \tau_k(\Lambda), \tag{6}$$

for any odd $h \leqq n$ and any even $g \leqq n$.

For example, we have for $h = 1$ and $h = 3$

$$\alpha(\Lambda) \geqq 1 - \rho(\Lambda) \quad \text{and} \quad \alpha(\Lambda) \geqq 1 - \overset{1}{\rho}_1(\Lambda) + \overset{2}{\rho}_2(\Lambda) - \overset{3}{\rho}_3(\Lambda) - \overset{2}{\rho}_3(\Lambda),$$

respectively. For the proof of Theorem 1 we need some lemmas. We consider the numbers

$$A_m^h = \sum_{k=0}^{h} \binom{m}{k} (-1)^k \qquad (0 \leqq h \leqq m, m > 0).$$

LEMMA 1.

$$A_m^h \leqq 0, \text{ if } h \text{ is odd};$$
$$A_m^h \geqq 0, \text{ if } h \text{ is even}.$$

PROOF OF LEMMA 1. We first assume $h<m/2$. Then we have

$$\binom{m}{r-1} \leqq \binom{m}{r},$$

if $r \leqq h$. Therefore, if h is odd, we see that

$$A_m^h = -\sum \begin{bmatrix} 1 \leqq r \leqq h \\ r \text{ odd} \end{bmatrix} \left\{ \binom{m}{r} - \binom{m}{r-1} \right\} \leqq 0;$$

and, if h is even,

$$A_m^h = 1 + \sum \begin{bmatrix} 1 \leqq r \leqq h \\ r \text{ even} \end{bmatrix} \left\{ \binom{m}{r} - \binom{m}{r-1} \right\} \geqq 0.$$

If $m>h \geqq m/2$, then $m-(h+1)<m/2$ and

$$A_m^h = \sum_{k=0}^{h} \binom{m}{k}(-1)^k = \sum_{k=0}^{m} \binom{m}{k}(-1)^k - \sum_{k=h+1}^{m} \binom{m}{k}(-1)^k$$
$$= 0 - \sum_{k=0}^{m-(h+1)} \binom{m}{k}(-1)^{m+k} = (-1)^{m+1} A_m^{m-(h+1)}.$$

Thus, if h is odd, we obtain the following:

If m is even, then $A_m^{m-(h+1)} \geqq 0$, $(-1)^{m+1} = -1$, and so $A_m^h \leqq 0$;
if m is odd, then $A_m^{m-(h+1)} \leqq 0$, $(-1)^{m+1} = 1$, and so $A_m^h \leqq 0$.

In a similar way we can prove that, if h is even, then $A_m^h \geqq 0$. If $m=h$, $A_m^m = 0$.

LEMMA 2. *Let $a_0, a_1, a_2, \cdots, a_m$ be real non-negative numbers, for which*

(7a) $$1 = a_0 = a_1, \qquad a_{2t} \geqq a_{2t+2} \qquad (0 \leqq 2t \leqq m-2)$$

and

(8a) $$a_{2t} \leqq a_{2t+1} \qquad (0 \leqq 2t \leqq m-1)$$

hold. Then we have

(9a) $$\sum_{k=0}^{h} \binom{m}{k}(-1)^k a_k \leqq 0,$$

if either h is odd and $h \leqq m$, or if $h = m$.

But if $b_0, b_1, b_2, \cdots, b_m$, are real non-negative numbers, for which

$$1 = b_0 = b_1, \qquad b_{2t-1} \geqq b_{2t+1} \qquad (2 \leqq 2t \leqq m-1) \tag{7b}$$

and

$$b_{2t-1} \leqq b_{2t} \qquad (2 \leqq 2t \leqq m) \tag{8b}$$

hold, then

$$\sum_{k=0}^{g} \binom{m}{k} (-1)^k b_k \geqq 0, \tag{9b}$$

if either g is even and $g \leqq m$, or if $g = m$.

Proof of Lemma 2. First we consider the case when (7a) and (8a) hold. We may assume that $a_{2t+1} = a_{2t}$. Then, using partial summation and Lemma 1, we have

$$\begin{aligned} \sum_{k=0}^{h} \binom{m}{k} (-1)^k a_k &= \sum \begin{bmatrix} 1 \leqq t \leqq h-1 \\ t \text{ odd} \end{bmatrix} (a_{t-1} - a_{t+1}) \sum_{k=0}^{t} \binom{m}{k} (-1)^k \\ &\quad + a_h \sum_{k=0}^{h} \binom{m}{k} (-1)^k \\ &\leqq a_h \sum_{k=0}^{h} \binom{m}{k} (-1)^k. \end{aligned}$$

Now the right side is less than or equal to 0, if h is odd, or if $h = m$. So (9a) is true. Similarly (7b) and (8b) imply (9b).

Lemma 3. *Let Λ be a lattice with $\rho(\Lambda) = m > 0$. We define numbers $a_0, a_1, a_2, \cdots, a_m$ and $b_0, b_1, b_2, \cdots, b_m$ by $a_0 = b_0 = 1$ and*

$$\tau_k(\Lambda) = a_k \binom{m}{k} \quad \text{and} \quad \pi_k(\Lambda) = b_k \binom{m}{k} \qquad (1 \leqq k \leqq m). \tag{10}$$

Now we assert the following: The a_k satisfy (7a) *and* (8a), *the b_k satisfy* (7b) *and* (8b).

Proof of Lemma 3. We have

$$\tau_1(\Lambda) = m = a_1 \binom{m}{1} = a_1 m$$

and therefore $a_1 = 1$. Defining constants c_k by

$$\overset{k}{\rho_k}(\Lambda) = c_k \binom{m}{k}$$

we obtain

$$c_{k+1}\binom{m}{k+1} = \overset{k+1}{\rho_{k+1}}(\Lambda)$$

$$= \{\text{the number of } (k+1)\text{-tuples } (X_1, \cdots, X_{k+1}) \text{ of lattice points of } \Lambda \text{ with } X_1 \in S, \cdots, X_{k+1} \in S \text{ of dimension } k+1\}$$

$$\leqq \overset{k}{\rho_k}(\Lambda)\,\frac{m-k}{k+1} = c_k\binom{m}{k}\frac{m-k}{k+1} = c_k\binom{m}{k+1}.$$

The inequality holds because each $(k+1)$-tuple considered can be represented as the union of a k-tuple of linearly independent points of Λ in S and another point of Λ in S in $k+1$ ways. But there are $\rho_k^k(\Lambda)$ such k-tuples and a k-tuple given, there are $m-k$ other points of Λ in S.

Dividing by

$$\binom{m}{k+1},$$

we obtain $c_{k+1} \leqq c_k$. Since, for even $k>0$, $a_k = c_k$, we have $a_{2t} \geqq a_{2t+2}$ for $t>0$. Also $a_0 = a_1 = c_1 \geqq c_2 = a_2$. Hence the a_k satisfy (7a). If $t>0$, then

$$a_{2t+1}\binom{m}{2t+1} = \tau_{2t+1}(\Lambda) = \overset{2t+1}{\rho_{2t+1}}(\Lambda) + \overset{2t}{\rho_{2t+1}}(\Lambda)$$

$$= \{\text{the number of } (2t+1)\text{-tuples } (X_1, \cdots, X_{2t+1}) \text{ of different lattice points of } \Lambda \text{ satisfying } X_1 \in S, \cdots, X_{2t+1} \in S \text{ of dimension } \geqq 2t\}$$

$$\geqq \overset{2t}{\rho_{2t}}(\Lambda)\,\frac{m-2t}{2t+1} = \tau_{2t}(\Lambda)\,\frac{m-2t}{2t+1} = a_{2t}\binom{m}{2t}\frac{m-2t}{2t+1} = a_{2t}\binom{m}{2t+1}.$$

Dividing by

$$\binom{m}{2t+1}$$

we obtain $a_{2t+1} \geqq a_{2t}$ and (8a).

If, in the above proof we replace a_k by b_k, τ_k by π_k, even by odd, and in places $2t+1$ by $2t$, then we obtain (7b) and (8b).

Proof of Theorem 1. Again let Λ be a lattice with $\rho(\Lambda) = m > 0$. Let the numbers a_k and b_k be defined by (10). Then the a_k satisfy (7a) and (8a), the b_k satisfy (7b) and (8b). If therefore h is odd, $h \leqq n$, $h \leqq m$, we have

$$1+\sum_{k=1}^{h}(-1)^k\tau_k(\Lambda) = \sum_{k=0}^{h}(-1)^k\binom{m}{k}a_k \leqq 0 = \alpha(\Lambda),$$

by Lemma 2. But if $h\leqq n$, $h\geqq m$, we obtain the same result:

$$1+\sum_{k=1}^{h}(-1)^k\tau_k(\Lambda) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}a_k \leqq 0 = \alpha(\Lambda).$$

In case g is even, $g\leqq n$, $g\leqq m$, we have

$$1+\sum_{k=1}^{g}(-1)^k\pi_k(\Lambda) = \sum_{k=0}^{g}(-1)^k\binom{m}{k}b_k \geqq 0 = \alpha(\Lambda);$$

and for $g\leqq n$, $g\geqq m$

$$1+\sum_{k=1}^{g}(-1)^k\pi_k(\Lambda) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}b_k \geqq 0 = \alpha(\Lambda).$$

Therefore Theorem 1 is true if $\rho(\Lambda)>0$. It is evidently true if $\rho(\Lambda)=0$.

2. We now calculate the integrals of $\rho_k^k(\Lambda)$ and $\rho_k^{k-1}(\Lambda)$ over the space of lattices with determinant 1.

THEOREM 2. *Suppose $k<n$. Then $\rho_k^k(\Lambda)$ is Borel-measurable in the space of lattices of determinant* 1 *and*

$$R_k^k = \int_F \rho_k^k(\Omega\Lambda_0)d\mu(\Omega) = \frac{1}{k!}V^k. \tag{11}$$

PROOF OF THEOREM 2. First, by the definition of $\rho_k^j(\Lambda)$, we see

$$\rho_k^j(\Lambda) = \frac{1}{k!}\sum\begin{bmatrix} X_1\in\Lambda,\cdots,X_k\in\Lambda \\ \dim(X_1,\cdots,X_k)=j \\ X_i\neq X_h, \text{ if } i\neq h\end{bmatrix}\rho(X_1)\cdots\rho(X_k), \tag{12}$$

where $\rho(X)$ is the characteristic function of S.

On the other hand, we observe the following theorem, stated by C. L. Siegel [**5**] and proved by C. A. Rogers[1] [**2**]: If

$$\psi(\Lambda) = \sum\begin{bmatrix} X_1\in\Lambda,\cdots,X_k\in\Lambda \\ \dim(X_1,\cdots,X_k)=k\end{bmatrix}\rho(X_1)\cdots\rho(X_k),$$

then

$$\int_F \psi(\Omega\Lambda_0)d\mu(\Omega)$$

[1] C. A. Rogers [**2**], Theorem 3, take $h=0$.

exists and is equal to

$$\int \cdots \int \rho(X_1) \cdots \rho(X_k) dX_1 \cdots dX_k.$$

Theorem 2 is an immediate consequence of these two results.

THEOREM 3. *Suppose $k<n$. Then $\rho_k^{k-1}(\Lambda)$ is Borel measurable in the space of lattices with determinant* 1, *and*

$$R_k^{k-1} = \int_F \rho_k^{k-1}(\Omega\Lambda_0)d\mu(\Omega)$$

$$(13) \qquad = \frac{1}{k!}\sum_{l=1}^{k}\sum_{q=1}^{\infty}\sum_{D}\frac{1}{q^n}\int \cdots \int \rho(X_1) \cdots \rho(X_{k-1})\rho\left(\sum_{i=1}^{k-1}\frac{d_i}{q}X_i\right)dX_1 \cdots dX_{k-1}.$$

Moreover,

$$(14) \qquad R_k^{k-1} \leqq \frac{V^{k-1}}{(k-1)!}\left[3^k(3/4)^{n/2} + 5^k 2^{-n}\right].$$

The sum in (13) *is over all integral vectors $D=(d_1, \cdots, d_{k-1})$, which have highest common factor relative prime to q, and which obey $|d_j|<q$ for $j<l$ and $|d_j| \leqq q$ for $j \geqq l$. Further, if $q=1$, D is not* $(0, 0, \cdots, 0)$ *nor of the form* $(0, \cdots, 0, 1, 0, \cdots, 0)$.

Before we can give a proof of Theorem 3 we need some lemmas.

LEMMA 4.

$$\sum\begin{bmatrix} X_1 \in \Lambda, \cdots, X_k \in \Lambda \\ \dim(X_1, \cdots, X_k) = k-1 \\ X_i \neq X_j \text{ if } i \neq j \end{bmatrix}\rho(X_1) \cdots \rho(X_k)$$

$$(15) \qquad = \sum_{l=1}^{k}\sum_{q=1}^{\infty}\sum_{D}\sum\begin{bmatrix} Y_1 \in \Lambda, \cdots, Y_{k-1} \in \Lambda \\ \dim(Y_1, \cdots, Y_{k-1}) = k-1 \\ \sum_{i=1}^{k-1} d_i/qY_i \in \Lambda \end{bmatrix}\rho(Y_1) \cdots \rho(Y_{k-1})\rho\left(\sum_{i=1}^{k-1}\frac{d_i}{q}Y_i\right),$$

where the sum on the right hand side is to be taken over the same set of vectors D as in Theorem 3.

PROOF OF LEMMA 4. If $X_1, \cdots, X_k$ is in the sum of the left hand side of (15), then $\dim(X_1, \cdots, X_k) = k-1$. Hence, the vectors $X_1, \cdots, X_k$ span $a(k-1)$-dimensional space. In this space we construct a system of orthogonal unit vectors $e_1, e_2, \cdots, e_{k-1}$. We write X_j in the form

$$X_j = \sum_{i=1}^{k-1} a_{ij} e_i \qquad (1 \leqq j \leqq k).$$

We define A_j $(1 \leqq j \leqq k)$ to be the determinant

$$\begin{vmatrix} a_{1\,1} \cdots a_{1\,j-1} & a_{1\,j+1} \cdots a_{1\,k} \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{k-1\,1} \cdots a_{k-1\,j-1} & a_{k-1\,j+1} \cdots a_{k-1\,k} \end{vmatrix}.$$

There exists a unique l, such that

$$|A_j| < |A_l|, \text{ if } j < l, \quad \text{and} \quad |A_j| \leqq |A_l|, \text{ if } j \geqq l.$$

This k-tuple $(X_1, \cdots, X_k)$ corresponds to the $(k-1)$-tuple $(Y_1, \cdots, Y_{k-1})$, defined by

$$Y_1 = X_1, \cdots, Y_{l-1} = X_{l-1},$$
$$Y_l = X_{l+1}, \cdots, Y_{k-1} = X_k,$$

and to the number l, to the vector $D = (d_1, \cdots, d_{k-1})$ and q, uniquely determined by

$$X_l = \sum_{i=1}^{k-1} \frac{d_i}{q} Y_i$$

and

$$\text{g.c.d.}\,(d_1, \cdots, d_{k-1}, q) = 1.$$

Because of our choice of l to make $|A_l|$ maximal we have

$$|d_t| < q, \text{ if } t < l, \quad \text{and} \quad |d_t| \leqq q, \text{ if } t \geqq l.$$

If $q = 1$, then $D(d_1, \cdots, d_{k-1})$ is not of the form $(0, 0, \cdots, 0)$ or $(0, \cdots, 0, 1, 0, \cdots, 0)$.

Since l, d, q, Y_j do not depend on any particular choice of the unit vectors $e_1, \cdots, e_{k-1}$, there corresponds to each term on the left side of (15) exactly one term on the right hand side. If, conversely, there are l, D, q, Y_j on the right side of (15), then we take the correspondence

$$X_1 = Y_1, \cdots, X_{l-1} = Y_{l-1}, \qquad X_l = \sum_{i=1}^{k-1} \frac{d_i}{q} Y_i,$$
$$X_{l+1} = Y_l, \cdots, X_k = Y_{k-1}.$$

These two mappings are one-one and inverse to each other. This proves the lemma.

LEMMA 5 (C. A. ROGERS). *Let $\rho(X_1, \cdots, X_m)$ be a Borel measurable function which is integrable in the Lebesgue sense over the whole $(X_1, \cdots, X_m)$-space. Let q be a positive integer and $D=(d_1, \cdots, d_m)$ be an integral vector with highest common factor relatively prime to q. Then the lattice function*

$$(16) \qquad \omega(\Lambda) = \sum \begin{bmatrix} X_1 \in \Lambda, \cdots, X_m \in \Lambda \\ \dim (X_1, \cdots, X_m) = m \\ \sum_{i=1}^{m} d_i/q, X_i \in \Lambda \end{bmatrix} \rho(X_1, \cdots, X_m)$$

is Borel measurable in the space of lattices of determinant 1, *and*

$$(17) \qquad \int_F \omega(\Omega\Lambda_0) d\mu(\Omega) = \frac{1}{q^n} \int \cdots \int \rho(X_1, \cdots, X_m) dX_1 \cdots dX_m.$$

PROOF OF LEMMA 5. Lemma 5 is essentially the case $h=1$ of Theorem 3 of C. A. Rogers [2]. The only difference is that we write $1/q$ instead of e_1/q as in *Rogers*, where $e_1=$g.c.d. (ϵ_1, q) and ϵ_1 is the elementary divisor of the matrix D. But since g.c.d. $(d_1, \cdots, d_m, q)=1$, we have $e_1=$g.c.d. $(\epsilon_1, q)=1$.

LEMMA 6 (C. A. ROGERS). *If $\rho(X)$ is a characteristic function, then*

$$(18) \qquad \int\int \rho(X)\rho(Y)\rho(X + Y + a) dX dY \leqq 2(3/4)^{n/2}\left(\int \rho(X) dX\right)^2.$$

PROOF OF LEMMA 6. See C. A. Rogers [3, Lemma 5].

PROOF OF THEOREM 3. (13) is a straightforward consequence of (12), Lemma 4 and Lemma 5 (take $m=k-1$). Therefore only (14) remains to be proved. (14) implies that both sides of (13) are finite. We evidently have

$$(19) \qquad R_k^{k-1} \leqq \frac{1}{k!} k \sum_{q=1}^{\infty} \sum_{D} \frac{1}{q^n} \int \cdots \int \rho(X_1) \cdots \rho(X_{k-1}) \rho\left(\sum_{i=1}^{k-1} \frac{d_i}{q} X_i\right) dX_1 \cdots dX_{k-1},$$

but now the summation is to be taken over all integral D with highest common factor relatively prime to q and $|d_j| \leqq q$. If $q=1$, then $D \neq (0, 0, \cdots, 0)$ and $\neq (0, \cdots, 0, 1, 0, \cdots, 0)$.

In (19) we mean that the inequality holds, if the right hand side is finite. We estimate the sum on the right hand side. We derive upper bounds (A) for the terms with $q=1$ and (B) for terms with $q>1$.

(A) There are $\leqq 3^{k-1}$ possibilities for D. D either has two elements d_i, d_j, both different from zero, or D is the form $(0, \cdots, 0, -1, 0, \cdots, 0)$. In the first case we have, by Lemma 6,

$$\int \cdots \int \rho(X_1) \cdots \rho(X_{k-1})\rho(\pm X_{i_1} \pm X_{i_2} \pm \cdots)dX_1 \cdots dX_{k-1}$$

$$\leqq 2(3/4)^{n/2}\left(\int \rho(X)dX\right)^{k-1} = 2(3/4)^{n/2}V^{k-1}.$$

If D is of the form $(0, \cdots, 0, -1, 0, \cdots, 0)$, then

$$\int \cdots \int \rho(X_1) \cdots \rho(X_{k-1})\rho(-X_i)dX_1 \cdots dX_{k-1} = 0.$$

Thus

$$\sum_D \frac{1}{1^n}\int \cdots \int \rho(X_1) \cdots \rho(X_{k-1})\rho\left(\sum_{i=1}^{k-1} \frac{d_i}{q} X_i\right)dX_1 \cdots dX_{k-1} \tag{20}$$
$$\leqq [3^k(3/4)^{n/2}]V^{k-1}.$$

(B) For a fixed $q>1$ the number of vectors D is at most $(2q+1)^{k-1} \leqq (5/2)^{k-1}q^{k-1}$. Consequently,

$$\begin{aligned} &\sum_{q=2}^{\infty}\sum_D \frac{1}{q^n}\int \cdots \int \rho(X_1) \cdots \rho(X_{k-1})\rho\left(\sum_{i=1}^{k-1} \frac{d_i}{q} X_i\right)dX_1 \cdots dX_{k-1} \\ &\qquad \leqq (5/2)^{k-1}\sum_{q=2}^{\infty} q^{k-1-n}V^{k-1} \leqq (5/2)^{k-1}2^{k+1-n}\sum_{q=2}^{\infty}\frac{1}{q^2}V^{k-1} \\ &\qquad < (5/2)^{k-1}2^{k+1-n}V^{k-1} < 5^k 2^{-n}V^{k-1}. \end{aligned} \tag{21}$$

By (19), (20) and (21) we get the upper bound

$$R_k^{k-1} \leqq \frac{1}{(k-1)!}[3^k(3/4)^{n/2} + 5^k 2^{-n}]V^{k-1}. \tag{14}$$

3. **Proof of Theorem 4.** Assume that (1) is satisfied. If h is odd and $h<n$, we infer from Theorem 1 that

$$\int_F \alpha(\Omega\Lambda_0)d\mu(\Omega) \geqq 1 + \sum_{k=1}^{h} (-1)^k \int_F \tau_k(\Omega\Lambda_0)d\mu(\Omega)$$

$$\geqq 1 + \sum_{k=1}^{h} (-1)^k R_k^k - \sum_{k=2}^{h} R_k^{k-1}$$

$$\geqq 1 + \sum_{k=1}^{h} (-1)^k \frac{V^k}{k!} - \sum_{k=2}^{h} [3^k(3/4)^{n/2} + 5^k 2^{-n}] \frac{V^{k-1}}{(k-1)!}.$$

Using the Taylor expansion of e^{-V} with a remainder after $h+1$ terms, we see that this implies that

$$\int_F \alpha(\Omega\Lambda_0)d\mu(\Omega)$$

$$\geqq e^{-V} - \sum_{k=2}^{h} [3^k(3/4)^{n/2} + 5^k 2^{-n}] \frac{V^{k-1}}{(k-1)!} - \frac{V^{h+1}}{(h+1)!}.$$

If g is even and $g<n$, we obtain, in a similar way

$$\int_F \alpha(\Omega\Lambda_0)d\mu(\Omega) \leqq e^{-V} + \sum_{k=2}^{g} [3^k(3/4)^{n/2} + 5^k 2^{-n}] \frac{V^{k-1}}{(k-1)!} + \frac{V^{g+1}}{(g+1)!}.$$

A combination of both these inequalities gives

$$m(A(S)) = \int_F \alpha(\Omega\Lambda_0)d\mu(\Omega) = e^{-V}(1 - R), \tag{2}$$

and

$$\begin{aligned} &-e^V\left[\sum_{k=2}^{g} [3^k(3/4)^{n/2} + 5^k 2^{-n}] \frac{V^{k-1}}{(k-1)!} + \frac{V^{g+1}}{(g+1)!}\right] \leqq R \\ &\leqq e^V\left[\sum_{k=2}^{h} [3^k(3/4)^{n/2} + 5^k 2^{-n}] \frac{V^{k-1}}{(k-1)!} + \frac{V^{h+1}}{(h+1)!}\right]. \end{aligned} \tag{22}$$

But, provided $1 \leqq k \leqq n$, we have

$$5^k 2^{-n} = 3^k(5/3)^k 2^{-n} < 3^k(5/6)^n < 3^k(3/4)^{n/2}. \tag{23}$$

So

$$\begin{aligned} &\sum_{k=2}^{h} [3^k(3/4)^{n/2} + 5^k 2^{-n}] \frac{V^{k-1}}{(k-1)!} e^V \\ &\qquad < 6(3/4)^{n/2} \sum_{k=2}^{h} \frac{3^{k-1}V^{k-1}}{(k-1)!} e^V < 6(3/4)^{n/2} e^{4V}. \end{aligned} \tag{24}$$

Now take h to be odd and to have either the value $n-1$ or the value $n-2$. Then as $V<n-1$ we have

$$\frac{V^{h+1}}{(h+1)!}e^V \leqq \frac{V^{n-1}}{(n-1)!}e^V.$$

Since

$$e^n > n^{n-1}/(n-1)!,$$

it follows that

$$\frac{V^{h+1}}{(h+1)!}e^V < V^{n-1}n^{-n+1}e^{V+n}. \tag{25}$$

Using (24) and (25) in (22) we obtain

$$R < 6(3/4)^{n/2}e^{4V} + V^{n-1}n^{-n+1}e^{V+n}.$$

A similar argument shows that

$$R > -6(3/4)^{n/2}e^{4V} - V^{n-1}n^{-n+1}e^{V+n}.$$

A combination of these inequalities gives (3) and proves Theorem 4.

THEOREM 5 (IMPROVEMENT OF THE MINKOWSKI-HLAWKA THEOREM). *Let S be a Borel set, not containing the origin* 0. *Suppose*

$$V \leqq \frac{1}{8} n \log 4/3 - \frac{1}{2} \log 3. \tag{26}$$

Then there exists an admissible lattice Λ *with determinant* 1.

In the original Minkowski-Hlawka Theorem there is $V<1$ instead of (26). It was first proved by E. Hlawka [**1**]. In the meantime it was proved to be true for $V<2/(1+2^{1-n})(1+3^{1-n})$ by the author [**4**] and for $V\leqq n^{1/2}/6$ if n is sufficiently large by C. A. Rogers [**3**].

PROOF OF THEOREM 5. We may assume that $X\in S$ implies $-X\notin S$. We may also assume $n\geqq 13$, because if $n<13$, then (26) yields $V<1$, and the theorem is true. (26) implies (1). Hence (2) and (3) hold. (26) also implies

$$6(3/4)^{n/2}e^{4V} \leqq 2/3.$$

Further, as $\log 4/3<1/3$, we have $V<n/24$. Also $e^{25/24}<24/23$. Thus

$$\begin{aligned} V^{n-1}n^{-n+1}e^{V+n} &< (1/24)^{n-1}e^{25n/24} \\ &< 24(24)^{-n}(24/23) \\ &= 24(23)^{-n} < 1/3. \end{aligned}$$

Combining these we obtain $|R|<1$, so that $m(A(S))>0$. Consequently, there exists an admissible lattice of determinant 1.

THEOREM 6. *Let S, T be two Borel sets. Assume that $X\in T$ yields $-X\notin S\cup T$ and that $0\notin S$. Further assume*

$$(27)\qquad \begin{aligned} V(S) &\leqq \frac{1}{16} n \log 4/3 - \frac{1}{2}\log 3 - 4(3/4)^{n/2}, \\ V(S\cup T) &\geqq V(S) + 4(3/4)^{n/4}. \end{aligned}$$

Then there exists a lattice Λ with determinant 1 *which is S-admissible, but not T-admissible.*

PROOF OF THEOREM 6. We may assume that $X\in S$ yields $-X\notin S$. Then never both $X\in S\cup T$ and $-X\in S\cup T$. We introduce $S_1=S$, $S_2=S\cup T$. We may assume that equality holds in the second equation (27), that is,

$$V(S_2) = V(S_1) + 4(3/4)^{n/4}.$$

Then

$$V(S_i) \leqq \frac{1}{16} n \log 4/3 - \frac{1}{2}\log 3.$$

Writing $\alpha_j(\Lambda)=\alpha_{S_j}(\Lambda)$, $V_j=V(S_j)$, $R_j=R(S_j)$, $c=(3/4)^{n/4}$, and applying Theorem 4 we infer

$$\int_F \alpha_i(\Omega\Lambda_0)d\mu(\Omega) = e^{-V_i}(1-R_i),$$

where

$$|R_i| \leqq \frac{2}{3}(3/4)^{n/4} + 24(23)^{-n} \leqq (3/4)^{n/4} = c < \frac{1}{2}\cdot$$

Hence

$$\begin{aligned} \int_F [\alpha_1(\Omega\Lambda_0) - \alpha_2(\Omega\Lambda_0)]d\mu(\Omega) &= e^{-V_1}(1-R_1) - e^{-V_2}(1-R_2) \\ &= e^{-V_2}\left[e^{V_2-V_1}(1-R_1) - (1-R_2)\right] \geqq e^{-V_2}\left[e^{4c}(1-c) - (1+c)\right] \\ &> e^{-V_2}\left[(1+4c)(1-c) - (1+c)\right] = e^{-V_2}(2c-4c^2) > 0. \end{aligned}$$

Consequently, there exists a lattice Λ satisfying $\alpha_1(\Lambda)-\alpha_2(\Lambda)>0$. This implies $\alpha_1(\Lambda)=1$, $\alpha_2(\Lambda)=0$. Therefore there is a point of Λ in $S_2=S\cup T$, but no point of Λ in $S_1=S$. Thus Λ is S-admissible, but not T-admissible.

THEOREM 7. *Let* $S_1, \cdots, S_m$ *be m Borel sets in* R_n, $n \geqq 13$, *each so that* $X \in S$ *yields* $-X \notin S$ *and with*

$$\sum_{j=1}^{m} e^{-W_j}[1 + R(n, V_j)] \leqq 1, \tag{28}$$

where $W_j = \min(V_j, n-1)$ *and* $R(n, V) = 6(3/4)^{n/2}e^{4V} + V^{n-1}n^{-n+1}e^{V+n}$. *Then there exists a lattice with determinant* 1 *which has at least one point in each* S_j.

PROOF OF THEOREM 7. Clearly it is enough to prove the theorem if $V_j \leqq n-1$. We obtain

$$\int_F \left[\sum_{j=1}^{m} \alpha_j(\Omega\Lambda_0) \right] d\mu(\Omega) < \sum_{j=1}^{m} e^{-V_j}[1 + R(n, V_j)] \leqq 1.$$

Consequently, there exists a lattice Λ such that $\sum_{j=1}^{m} \alpha_j(\Lambda) = 0$ and Λ is not admissible for any S_j.

REFERENCES

1. E. Hlawka, *Zur Geometrie der Zahlen*, Math. Zeit. vol. 49 (1944) pp. 285–312.

2. C. A. Rogers, *Mean values over the space of lattices*, Acta Math. vol. 94 (1955) pp. 249–287.

3. ———, *The number of lattice points in a set*, Proc. London Math. Soc. vol. 22 (1956) pp. 305–320.

4. W. Schmidt, *Eine Verschärfung des Satzes von Minkowski-Hlawka*. Monatshefte für Mathematik vol. 60 (1956) pp. 110–113.

5. C. L. Siegel, *A mean value theorem in geometry of numbers*, Ann. of Math. vol. 46 (1945) pp. 340–347.

MONTANA STATE UNIVERSITY