ON PROPERTIES CHARACTERIZING PSEUDO-COMPACT SPACES

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Completely regular pseudo-compact spaces have been characterized in several ways. E. Hewitt [**6**, pp. 68–70] has given one characterization in terms of the Stone-Čech compactification and another in terms of the zero sets of continuous functions. J. Colmez [**2**; no proofs included] and I. Glicksberg [**4**] have obtained characterizations by means of a convergence property for sequences of continuous functions (Dini's theorem) and in terms of sequences of closed neighborhoods. A very elegant characterization in terms of a covering property has been obtained by S. Mardešić and P. Papić [**8**]. This characterization is discussed by K. Iseki and S. Kasahara in [**7**].[1] In this paper we obtain an additional characterization of pseudo-compact spaces by means of a convergence property for sequences of continuous functions and characterize completely regular pseudo-compact spaces by a covering property. In Theorem 1 we establish the equivalence of many of the topological characterizations of pseudo-compactness (for completely regular spaces) in arbitrary topological spaces. Among the applications of these results is a theorem concerning products of pseudo-compact spaces. We omit strong separation axoms in the definitions of some common terms; e.g., a space X (not necessarily completely regular) is *pseudo-compact* [**6**] if every continuous real-valued function on X is bounded, and a space X (not necessarily Hausdorff) is *paracompact* [**9**] if every open covering has a locally finite refinement.

DEFINITION. A topological space X is *lightly compact* if every locally finite collection of open sets of X is finite.

A topological space is countably compact if every countable open covering has a finite subcovering. We will see (Theorem 1) that a countably compact space is lightly compact. There are examples [**6**, p. 69] of lightly compact, locally compact, completely regular T_1 spaces which are not countably compact.

THEOREM 1. *If X is a topological space, the following are equivalent:*

(i) *X is lightly compact.*

(ii) *Every countable, locally finite, disjoint collection of open sets of X is finite.* (*See* [**8**].)

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[1] The authors wish to thank the referee for this reference.

(iii) *If* $\mathfrak{U}$ *is a countable open covering of* X *and* A *is an infinite subset of* X, *then the closure of some member of* $\mathfrak{U}$ *contains infinitely many points of* A.

(iv) *If* $\mathfrak{U}$ *is a countable open covering of* X, *then there is a finite subcollection of* $\mathfrak{U}$ *whose closures cover* X. (*See* [2].)

PROOF. That (i) implies (ii) and (iii) implies (iv) is obvious. If (iii) is false, then there is a countable open covering $\{U_n\}$ and a countably infinite set $\{x_j\}$ such that, for each n, $\overline{U}_n \cap \{x_j\}$ is finite. Let y_1 be the first point of $\{x_j\}$ which is not in $\overline{U}_1$. Put $V_1 = U_{n_1} - \overline{U}_1$ where U_{n_1} is the first set in $\{U_n\}$ which contains y_1. Let y_{k+1} be the first point of $\{x_j\}$ which is not in $\bigcup_{i=1}^{n_k} \overline{U}_i$. Put $V_{k+1} = U_{n_{k+1}} - \bigcup_{i=1}^{n_k} \overline{U}_i$, where $U_{n_{k+1}}$ is the first set in $\{U_n\}$ which contains y_{k+1}. Since $\{U_n\}$ is a covering of X, the collection $\{V_k\}$ is an infinite, locally finite disjoint collection. Hence, (ii) implies (iii). For the final implication let $\mathfrak{U}$ be an infinite locally finite collection of open sets. Let $\{U_n\}$ be a countably infinite subcollection of $\mathfrak{U}$. Since $\{U_n\}$ is locally finite, for each positive integer j, the set $X - \bigcup_{n \geqq j} \overline{U}_n$ is open, and $\{X - \bigcup_{n \geqq j} \overline{U}_n\}$ covers X. Clearly, no finite subcollection of $\{\mathrm{Cl}(X - \bigcup_{n \geqq j} \overline{U}_n)\}$ covers X since this sequence is monotone increasing, and $\mathrm{Cl}\,(X - \bigcup_{n \geqq j} \overline{U}_n)$ does not contain U_k if $k \geqq j$. This contradicts (iv). Hence, (iv) implies (i).

We say that a sequence $\{f_n\}$ of real valued functions converges to a function f *uniformly at* $x \in X$ if the following is true: If ϵ is a positive number, then there is a neighborhood U_x of x and a positive number N such that $|f_n(y) - f(y)| < \epsilon$ whenever $y \in U_x$ and $n \geqq N$. A sequence $\{f_n\}$ converges to f *locally uniformly* when each point has a neighborhood on which $\{f_n\}$ converges to f uniformly.

THEOREM 2. *If* X *is a topological space, the following are equivalent*:

(i) X *is pseudo-compact.*

(ii) *Every sequence of continuous functions which converges uniformly at each point of* X *converges uniformly on* X.

(iii) *Every sequence of continuous functions which converges locally uniformly on* X *converges uniformly on* X.

PROOF. It is obvious that (ii) implies (iii). Suppose that X does not satisfy (ii). Then there is a sequence $\{f_n\}$ of continuous functions which converges to a function f uniformly at each point of X, a sequence $\{x_n\}$ of points of X, and a positive number ϵ such that $|f_n(x_n) - f(x_n)| > \epsilon$. By virtue of the pointwise uniform convergence, the sequence $\{x_n\}$ can be so chosen that $x_i \neq x_j$ when $i \neq j$. Let

$$g_n(x) = \max\,(|f_n(x) - f(x)| - \epsilon, 0)$$

and

$$g(x) = \sum_{n=1}^{\infty} \frac{n}{g_n(x_n)} g_n(x).$$

The function $g(x)$ is well-defined and continuous since, by uniform convergence at points, there is a neighborhood of each point of X in which all but finitely many of the functions g_n vanish. Moreover, since $g_n(x_n) \geqq n$, g is unbounded. This shows that (i) implies (ii). Suppose that X is not pseudo-compact. Then there is a positive unbounded continuous function g on X. Let $g_n(x) = g(x)$ for all x such that $g(x) \leqq n$ and let $g_n(x) = n$ for all x such that $g(x) \geqq n$. Clearly, each g_n is continuous, and the sequence $\{g_n\}$ converges to g locally uniformly but not uniformly. This shows that (iii) implies (i).

THEOREM 3. *A completely regular space is pseudo-compact if and only if it is lightly compact.*

PROOF. Any topological space which is lightly compact is also pseudo-compact. If a completely regular space X is not lightly compact, then there is a countably infinite locally finite collection $\{U_n\}$ of non empty open sets of X. If $x_n \in U_n$, there is a function f_n such that $f_n(x_n) = 1$, $f_n(x) = 0$ for $x \notin U_n$, and $0 \leqq f_n(x) \leqq 1$ for all $x \in X$. Since $\{U_n\}$ is locally finite, the function $\sum_{n=1}^{\infty} nf_n$ is real and continuous. Also, $\sum_{n=1}^{\infty} nf_n(x_n) \geqq n$, so that X is not pseudo-compact.

THEOREM 4. *If X is a normal T_1 space, the following are equivalent:*
(i) *X is pseudo-compact.*
(ii) *X is lightly compact.*
(iii) *X is countably compact.*

PROOF. Hewitt [**6**] has shown the equivalence of (i) and (iii). That (i) and (ii) are equivalent follows immediately from Theorem 3.

The following is an example of a pseudo-compact T_1 space which is not lightly compact.

Example: Let $$X = \{x_{ij}\} \cup (\textstyle\bigcup_{ij} Y_{ij}), \qquad (i, j = 1, 2, \cdots),$$

where each x_{ij} is a point and each Y_{ij} is an infinite set. For a neighborhood of a point x_{ij} we take the set consisting of x_{ij} plus all but a finite number of points of each of the sets Y_{kj}, $(k \geqq i)$ and Y_{ik}, $(k \geqq j)$. For neighborhoods of points in Y_{ij} we take complements (relative to Y_{ij}) of finite sets.

To see that X is not lightly compact we take the covering by sets.

$$U_{ij} = (x_{ij}) \cup \left(\bigcup_{k \geqq i} Y_{kj} \right) \cup \left(\bigcup_{k \geqq j} Y_{ik} \right).$$

Each $\overline{U}_{ij}$ contains only finitely many points of $\{x_{ii}\}$. Thus $\{\overline{U}_{ij}\}$ does not contain a finite subcover. Each real valued continuous function on X is constant. Thus X is pseudo-compact. Simpler examples of pseudo-compact T_1 spaces which are not lightly compact exist. This example was chosen because of the following property of X: If $\{U_n\}$ is a countable open covering of X then $\{\overline{U}_n\}$ contains a proper subcovering. This property implies pseudo-compactness in any topological space.

Theorem 3 may be applied to obtain a simple proof that the topological product of two completely regular pseudo-compact spaces is pseudo-compact if at least one of the spaces is first countable (cf. [5]).[2] We first prove two results (Theorems 5 and 6) concerning the product of lightly compact spaces.

THEOREM 5. *Let X and Y be lightly compact and let X satisfy the following property: For each infinite collection $\mathfrak{V}$ of open sets of X which is not locally finite there is a point $x\in X$ and an infinite subcollection $\mathfrak{U}\subset\mathfrak{V}$ such that each neighborhood of x intersects all but finitely many elements of $\mathfrak{U}$. Then $X\times Y$ is lightly compact.*

PROOF. Two easily proved preliminary statements about collections of rectangular open sets are needed. First, if X is a lightly compact space and $\{U_\alpha\times V_\alpha\}$ is an infinite locally finite collection of rectangular open sets of $X\times Y$, then $\{V_\alpha\}$ is infinite. Second, if $\{W_\alpha\}$ is an infinite locally finite collection of subsets of $X\times Y$ and if the nonempty sets U_α and V_α are such that $U_\alpha\times V_\alpha\subset W_\alpha$ for each α, then $\{U_\alpha\times V_\alpha\}$ is both infinite and locally finite. Now we suppose that X and Y satisfy the hypothesis of the theorem, but that $X\times Y$ is not lightly compact. Then, according to our second preliminary comment, there exists an infinite locally finite collection $\{U_\alpha\times V_\alpha\}$ of rectangular open sets of $X\times Y$. According to the first preliminary statement, $\{U_\alpha\}$ cannot be finite. Hence, by the hypothesis, there is a point x of X and an infinite subcollection $\{U_\beta\}$ of $\{U_\alpha\}$ such that each neighborhood of x intersects all but finitely many U_β. The collection $\{U_\beta\times V_\beta\}$ is locally finite since $\{U_\alpha\times V_\alpha\}$ is locally finite, and, again by the first preliminary statement, $\{V_\beta\}$ is infinite. Let y be a point at which $\{V_\beta\}$ is not locally finite. If $U\times V$ is any rectangular open set such that $(x, y)\in U\times V$, then clearly $U\times V\cap U_\beta\times V_\beta\neq 0$ for infinitely many $U_\beta\times V_\beta$, which contradicts the local finiteness of $\{U_\alpha\times V_\alpha\}$.

LEMMA. *Let X be first countable, $\mathfrak{V}$ a collection of subsets of X and x*

[2] The authors wish to thank Professor Melvin Henriksen, who proposed this problem in communicating to them the results of [5].

a point at which $\mathfrak{V}$ *is not locally finite. Then there is an infinite subcollection* $\mathfrak{U}\subset\mathfrak{V}$ *such that every neighborhood of* x *intersects all but finitely many elements of* $\mathfrak{U}$.

PROOF. Let $\{G_k\}$ be a monotone decreasing base at x. If $\mathfrak{U}_k$ is the set of elements of $\mathfrak{V}$ intersecting G_k, then $\mathfrak{U}_{k+1}\subset\mathfrak{U}_k$. Since each $\mathfrak{U}_k$ is infinite, we may choose $\mathfrak{U}=\{U_1, U_2, \cdots\}$, where $U_k\in\mathfrak{U}_k$ and $U_k\neq U_j$ for $j\neq k$. Then $G_k\cap U_n\neq 0$ for $n\geqq k$.

THEOREM 6. *If* X *is first countable, then* $X\times Y$ *is lightly compact if and only if both* X *and* Y *are lightly compact.*

PROOF. According to the lemma, the hypothesis of Theorem 5 is satisfied when X is first countable and lightly compact and Y is lightly compact. The converse is obvious.

COROLLARY. *If* X *and* Y *are completely regular and* X *is first countable, then* $X\times Y$ *is pseudo-compact if and only if both* X *and* Y *are pseudo-compact.*

A space is *metacompact* [1] if every open covering has a point-finite refinement. A space X has the *Michael property* [**9**; property (*)] if every open covering of X has a refinement which is the union of countably many locally finite collections of open sets.

REMARK. *For lightly compact spaces, the Lindelöf property is equivalent to the Michael property.*

THEOREM 7. *If* X *is a topological space, then* X *is compact if and only if* X *has the Michael property and is countably compact.*

PROOF. Since a countably compact space is lightly compact, the Michael property implies the Lindelöf property by the above Remark. Thus, a countably compact space with the Michael property is compact. The reverse implication is obvious.

R. Arens and J. Dugundji [1] have proved that compactness is equivalent to metacompactness plus countable compactness. In a T_0 space the Michael property does not imply metacompactness. For example, let $X=\{x\,|\,0<x\leqq 1\}$. A neighborhood of x is a set $\{y\,|\,x-\epsilon<y\leqq 1\}$. However, E. Michael has shown that the Michael property is equivalent to paracompactness in a regular space [**9**].

The next two theorems are proved easily and will be stated without proof.

THEOREM 8. *A topological space is compact if and only if it is paracompact and lightly compact.*

THEOREM 9. *A topological space is countably compact if and only if it is countably paracompact and lightly compact.*

THEOREM 10. *A regular (completely regular) space is compact if and only if it is lightly compact (pseudo-compact) and has the Michael property.*

PROOF. This follows immediately from Theorems 3 and 8 and the equivalence of paracompactness to the Michael property in a regular space.

THEOREM 11. *A regular T_1 space is compact if and only if it is pseudo-compact and has the Michael property.*

PROOF. The theorem follows immediately from Theorems 3 and 8 since a regular T_1 space with the Michael property is paracompact and T_2 and thus normal and T_2.

Using the fact that a metric space is regular and T_1 and has the Michael property, we have an alternate proof that in a metric space pseudo-compactness is equivalent to compactness [**3**, p. 196].

THEOREM 12. *A T_1 space X is compact if and only if it is lightly compact, has the Michael property and each point of X has a neighborhood whose boundary is countably compact.*

PROOF. Suppose that X is not compact and has all the other properties of the theorem. It is sufficient to prove that every infinite set has a point of accumulation, since X is T_1 and has the Lindelöf property, as the Remark above shows. Suppose that $\{x_j\}$ is an infinite set which has no point of accumulation. For each point x there is a neighborhood U_x such that U_x contains at most a finite number of points of $\{x_j\}$ and $\overline{U}_x - U_x$ is countably compact. Thus, $\overline{U}_x - U_x$ contains at most a finite number of points of $\{x_j\}$. The covering $\{U_x\}$ has a countable subcovering because of the Lindelöf property. This contradicts the assumption that X is lightly compact, using equivalence of (i) and (iv) of Theorem 1. The other implication is obvious.

K. Morita [**10**] has shown the following: If X is a Hausdorff space and each neighborhood V_x in X contains a neighborhood U_x such that $\overline{U}_x - U_x$ is compact, then X is completely regular. This result with Theorems 3 and 11 yields the following:

THEOREM 13. *A Hausdorff space is compact if and only if it is pseudo-compact, has the Michael property and each neighborhood V_x contains a neighborhood U_x whose boundary is compact.*

A closed subset of a lightly compact space need not be lightly compact. This follows from the fact that there are lightly compact spaces which are not countably compact. In regard to this we have the following:

THEOREM 14. *A topological space is lightly compact if and only if every proper subset which is the closure of an open set is lightly compact.*

PROOF. Suppose X is lightly compact and that $\{U_n \cap \overline{U}\}$ covers $\overline{U}$, where U is an open subset of X. Then $\{U_n, X-\overline{U}\}$ covers X. Thus, $\{\overline{U}_n, \mathrm{Cl}\,(X-\overline{U})\}$ has a finite subcovering. Let $\overline{U}_{n_1}, \cdots, \overline{U}_{n_k}$, $\mathrm{Cl}(X-\overline{U})$ cover X. Then $\{\overline{U}_{n_i}\}_1^k$ covers U. But $\overline{U}_{n_i} \cap U \subset \mathrm{Cl}(U_{n_i} \cap \overline{U})$. Therefore, $\{\mathrm{Cl}(U_{n_i} \cap \overline{U})\}_{i=1}^k$ covers $\overline{U}$. Thus $\overline{U}$ is lightly compact. For the reverse implication let $\{U_n\}$ cover X. If $\overline{U}_1$ does not cover X, then $\mathrm{Cl}\,(X-\overline{U}_1)$, is a proper subset and, consequently, is lightly compact. Since $\{U_n\}_{n=2}^{\infty}$ covers $\mathrm{Cl}\,(X-\overline{U}_1)$, a finite number of the closures cover $\mathrm{Cl}\,(X-\overline{U}_1)$. These closures with $\overline{U}_1$ cover X. This shows that X is lightly compact.

REFERENCES

1. R. Arens and J. Dugundji, *Remark on the concept of compactness*, Portugaliae Mathematica vol. 9 (1950) pp. 141–143.

2. J. Colmez, *Sur les espaces précompacts*, C. R. Acad. Sci. Paris vol. 233 (1951) pp. 1552–1553.

3. E. Correl and M. Henriksen, *On rings of bounded continuous functions with values in a division ring*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 194–198.

4. I. Glicksberg, *The representation of functionals by integrals*, Duke Math. J. vol. 19 (1952) pp. 253–261.

5. M. Henriksen and J. R. Isbell, *On the Stone-Čech compactification of a product of two spaces*, Bull. Amer. Math. Soc. Abstract 63-2-332.

6. E. Hewitt, *Rings of real-valued continuous functions.* I, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99.

7. K. Iseki and S. Kasahara, *On pseudo-compact and countably compact spaces*, Proc. Japan Acad. vol. 33 (1957) pp. 100–102.

8. S. Mardešić and P. Papić, *Sur les espaces dont toute transformation réelle continue est bornée*, Hrvatsko Prirod. Drustvo. Glasnik Mat.-Fiz. Astr. Ser. II. vol. 10 (1955) pp. 225–232.

9. E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 831–838.

10. K. Morita, *On bicompactification of semi-bicompact spaces*, Science Reports of the Tokyo Bunrika Daigaku. Sect. A vol. 4 (1952) pp. 222–229.

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