ON A THEOREM OF MONTGOMERY AND SAMELSON

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In their paper on fibrations with singularities Montgomery and Samelson showed that if a compact connected Lie group acts differentiably and effectively on a sphere and if there is one stationary point, then the remaining orbits cannot all be of the same dimension [**2**]. In our note we will examine transformation groups in which all orbits have the same dimension. As a corollary, we shall extend the Montgomery-Samelson theorem to all closed simply connected manifolds. We shall not use a differentiability hypothesis.

We shall denote by (L, X) the action of a compact, connected Lie group L on a locally compact connected separable metric space X. It will be assumed that all the orbits of (L, X) have the same dimension. We denote by

$$m: L \times X \to X$$

the function defining the action of L on X, and by

$$\eta: X \to X/L$$

the natural map of X onto the orbit space X/L. Let G_x, H_x and N_x respectively denote the isotropy group at x, the identity component of G_x, and the normalizer in L of H_x. Since all the orbits in X have the same dimension, $\dim H_x = \dim H_y$. If x and y are sufficiently close there is an element $g \in L$ such that $gG_xg^{-1} \subset G_y$ [**3**, p. 215]. Now $\dim H_x = \dim H_y$, so we must have $gH_xg^{-1} = H_y$. Since X is connected, this implies that all the H_x are conjugate.

Let $F_x \subset X$ be the set of stationary points of H_x. We first observe that if $F_x \cap F_y \neq \phi$, then $H_x = H_y$, for if not, then the points of the intersection are stationary under the closed subgroup generated by H_x and H_y, but the identity component of this subgroup will contain $H_x \cup H_y$ so that it cannot possibly be conjugate to H_x. If $g \in L$ is such that $gF_x = F_x$, then $g \in N_x$, for gF_x is the set of stationary points of gH_xg^{-1}.

LEMMA 1. *For any $x \in X$, the orbit space F_x/N_x is homeomorphic to X/L.*

We may use the map η to define a map

$$\tilde{\eta}: F_x/N_x \to X/L.$$

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The map is onto, for if $O(y) \subset X$ is the orbit of y, then $gy \in F_x$, where $g \in L$ is such that $gH_yg^{-1} = H_x$. The map is 1-1 since $gF_x = F_x$ implies that $g \in N_x$. We omit the proof that the inverse is continuous.

Let $\tilde{F}_x$ be a component of F_x and let $B_x \subset N_x$ be the subgroup of L which maps $\tilde{F}_x$ onto itself. We define two mappings

$$\nu_1: X \to L/B_x,$$

$$\nu_2: X \to L/N_x.$$

The first map is defined by sending the points of $g\tilde{F}_x$ into the coset gB_x, and the second by sending points of gF_x into the coset gN_x.

Lemma 2. *The mappings ν_1 and ν_2 define local product bundles with fibre and group $\tilde{F}_x$, B_x and F_x, N_x respectively.*

It is well known that

$$\nu: L \to L/B_x$$

is a principal bundle. Let $J \subset L$ be a local cross-section at the identity e of ν. The cell J has the property that if g, $h \in J$ and $g^{-1}h \in B_x$, then $g^{-1}h = e$.

The map

$$m: J \times \tilde{F}_x \to X$$

is a homeomorphism, for if $gy = hz$, then $g^{-1}hz = y$, and $g^{-1}h \in B_x$, which implies that $g = h$ and $y = z$. Of course $\nu_1(J\tilde{F}_x) = \nu(J)$. The translates of J define a set of neighborhoods on L/B_x which serve at once as co-ordinate neighborhoods for $[L, L/B_x, B_x; \nu]$ and for $[X, L/B_x, \tilde{F}_x; \nu_1]$, and this determines the co-ordinate transformations of $[X, L/B_x, \tilde{F}_x; \nu_1]$. A similar statement applies to $[X, L/N_x, F_x; \nu_2]$.

It should be observed that if $y \in \tilde{F}_x$, then $G_y \subset B_x$. It might be noted in passing that some local properties of $\tilde{F}_x$ can be determined now. For example, if X is an ANR, so is $\tilde{F}_x$, or if X is a manifold, $\tilde{F}_x$ is a generalized manifold in the sense of local homology properties.

Lemma 3. *If for the transformation group (L, X) the space X is simply connected, and if all the orbits have the same dimension, then B_x is connected.*

The proof is immediate upon examining the fiber bundle $[X, L/B_x, \tilde{F}_x; \nu_1]$. Since X is simply connected, $\pi_1(L/B_x) = 0$, but on the other hand, there is a homomorphism of $\pi_1(L/B_x)$ onto B_x/B_x^c, where B_x^c is the identity component of B_x.

Theorem 1. *If (L, X) denotes the action of a compact, connected Lie* *group on a compact simply connected ANR such that $\chi(X)\neq 0$, and if all the orbits have the same dimension, then all the isotropy groups are connected and conjugate.*

We may take L to be simply connected. Since $\chi(X)\neq 0$, any maximal torus in L has a stationary point in X, hence H_x is a subgroup of maximum rank. By the preceding lemma B_x is connected and $H_x\subset B_x\subset N_x$, but N_x/H_x is finite, so $B_x=H_x$. Finally, since $G_x\subset B_x$, $G_x=H_x$. This means that L defines a fibration of X with fibre L/H_x [3].

THEOREM 2. *If (L, M^n) denotes the effective action of a compact, connected Lie group on a closed simply connected manifold, and if there is a stationary point, then the remaining orbits cannot all have the same dimension.*

Let us suppose that the theorem is false, and that (L, M^n) is a counter example. We consider the transformation group (L, W^n) on the open simply connected manifold W^n obtained by deleting the stationary point from M^n. We assume that the orbits of (L, W^n) all have the same dimension. It was shown in [1] that any maximal torus in L has a stationary point on W^n, thus H_x has maximal rank. Furthermore $\pi_1(L/N_x)\simeq N_x/H_x$, so that F_x has (N_x/H_x) components. Since N_x must act transitively on the components of F_x, it follows that $B_x=H_x$. This implies that $G_x=H_x$, so all the isotrophy groups are connected and conjugate. Thus (L, W^n) defines a compact fibration of W^n by L/H_x which must be proper, and this contradicts [1].[1]

This is the extension of the Montgomery-Samelson theorem referred to in our title. It should be kept in mind that there is a conjecture to the effect that a compact connected Lie group acting on a sphere with one stationary point must have another. The Montgomery-Samelson theorem supports this conjecture and is a step toward a solution.

REFERENCES

1. P. E. Conner, *On the impossibility of fibering certain manifolds by a compact fiber*, Michigan Mathematical Journal (to appear).

2. D. Montgomery and H. Samelson, *Fiberings with singularities*, Duke Math. J. vol. 13 (1946) pp. 51–56.

3. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, 1955.

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[1] The principal result of [1] asserts that an open simply connected manifold whose one-point compactification is again a manifold cannot be fibred by a compact fibre.