ON A THEOREM OF MONTGOMERY AND SAMELSON

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In their paper on fibrations with singularities Montgomery and Samelson showed that if a compact connected Lie group acts differentiably and effectively on a sphere and if there is one stationary point, then the remaining orbits cannot all be of the same dimension [2]. In our note we will examine transformation groups in which all orbits have the same dimension. As a corollary, we shall extend the Montgomery-Samelson theorem to all closed simply connected manifolds. We shall not use a differentiability hypothesis.

We shall denote by \((L, X)\) the action of a compact, connected Lie group \(L\) on a locally compact connected separable metric space \(X\). It will be assumed that all the orbits of \((L, X)\) have the same dimension. We denote by

\[
m: L \times X \to X
\]

the function defining the action of \(L\) on \(X\), and by

\[
\eta: X \to X/L
\]

the natural map of \(X\) onto the orbit space \(X/L\). Let \(G_x\), \(H_x\) and \(N_x\) respectively denote the isotropy group at \(x\), the identity component of \(G_x\), and the normalizer in \(L\) of \(H_x\). Since all the orbits in \(X\) have the same dimension, \(\dim H_x = \dim H_y\). If \(x\) and \(y\) are sufficiently close there is an element \(g \in L\) such that \(gG_xg^{-1} \subseteq G_y\) [3, p. 215]. Now \(\dim H_x = \dim H_y\), so we must have \(gH_xg^{-1} = H_y\). Since \(X\) is connected, this implies that all the \(H_x\) are conjugate.

Let \(F_x \subseteq X\) be the set of stationary points of \(H_x\). We first observe that if \(F_x \cap F_y \neq \emptyset\), then \(H_x = H_y\), for if not, then the points of the intersection are stationary under the closed subgroup generated by \(H_x\) and \(H_y\), but the identity component of this subgroup will contain \(H_x \cup H_y\) so that it cannot possibly be conjugate to \(H_x\). If \(g \in L\) is such that \(gF_x = F_x\), then \(g \in N_x\), for \(gF_x\) is the set of stationary points of \(gH_xg^{-1}\).

**Lemma 1.** For any \(x \in X\), the orbit space \(F_x/N_x\) is homeomorphic to \(X/L\).

We may use the map \(\eta\) to define a map

\[
\dot{\eta}: F_x/N_x \to X/L.
\]

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The map is onto, for if $O(y) \subset X$ is the orbit of $y$, then $gy \in F_x$, where $g \in L$ is such that $gH_0g^{-1} = H_x$. The map is 1-1 since $gF_x = F_x$ implies that $g \in N_x$. We omit the proof that the inverse is continuous.

Let $\tilde{F}_x$ be a component of $F_x$ and let $B_x \subset N_x$ be the subgroup of $L$ which maps $\tilde{F}_x$ onto itself. We define two mappings

$$v_1: X \to L/B_x,$$
$$v_2: X \to L/N_x.$$  

The first map is defined by sending the points of $g\tilde{F}_x$ into the coset $gB_x$, and the second by sending points of $gF_x$ into the coset $gN_x$.

**Lemma 2.** The mappings $v_1$ and $v_2$ define local product bundles with fibre and group $\tilde{F}_x$, $B_x$ and $F_x$, $N_x$ respectively.

It is well known that

$$v: L \to L/B_x$$

is a principal bundle. Let $J \subset L$ be a local cross-section at the identity $e$ of $v$. The cell $J$ has the property that if $g, h \in J$ and $g^{-1}h \in B_x$, then $g^{-1}h = e$.

The map

$$m: J \times \tilde{F}_x \to X$$

is a homeomorphism, for if $gy = hz$, then $g^{-1}hz = y$, and $g^{-1}h \in B_x$, which implies that $g = h$ and $y = z$. Of course $v_1(J\tilde{F}_x) = v(J)$. The translates of $J$ define a set of neighborhoods on $L/B_x$ which serve at once as co-ordinate neighborhoods for $[L, L/B_x, B_x; v]$ and for $[X, L/B_x, \tilde{F}_x; v_1]$, and this determines the co-ordinate transformations of $[X, L/B_x, \tilde{F}_x; v_1]$. A similar statement applies to $[X, L/N_x, F_x; v_2]$.

It should be observed that if $y \in \tilde{F}_x$, then $G_y \subset B_x$. It might be noted in passing that some local properties of $\tilde{F}_x$ can be determined now. For example, if $X$ is an ANR, so is $\tilde{F}_x$, or if $X$ is a manifold, $\tilde{F}_x$ is a generalized manifold in the sense of local homology properties.

**Lemma 3.** If for the transformation group $(L, X)$ the space $X$ is simply connected, and if all the orbits have the same dimension, then $B_x$ is connected.

The proof is immediate upon examining the fiber bundle $[X, L/B_x, \tilde{F}_x; v_1]$. Since $X$ is simply connected, $\pi_1(L/B_x) = 0$, but on the other hand, there is a homomorphism of $\pi_1(L/B_x)$ onto $B_x/B'_x$, where $B'_x$ is the identity component of $B_x$.

**Theorem 1.** If $(L, X)$ denotes the action of a compact, connected Lie
group on a compact simply connected ANR such that \( \chi(X) \neq 0 \), and if all the orbits have the same dimension, then all the isotropy groups are connected and conjugate.

We may take \( L \) to be simply connected. Since \( \chi(X) \neq 0 \), any maximal torus in \( L \) has a stationary point in \( X \), hence \( H_x \) is a subgroup of maximum rank. By the preceding lemma \( B_x \) is connected and \( H_x \subset B_x \subset N_x \), but \( N_x/H_x \) is finite, so \( B_x = H_x \). Finally, since \( G_x \subset B_x \), \( G_x = H_x \). This means that \( L \) defines a fibration of \( X \) with fibre \( L/H_x \) [3].

**Theorem 2.** If \((L, M^n)\) denotes the effective action of a compact, connected Lie group on a closed simply connected manifold, and if there is a stationary point, then the remaining orbits cannot all have the same dimension.

Let us suppose that the theorem is false, and that \((L, M^n)\) is a counter example. We consider the transformation group \((L, W^n)\) on the open simply connected manifold \( W^n \) obtained by deleting the stationary point from \( M^n \). We assume that the orbits of \((L, W^n)\) all have the same dimension. It was shown in [1] that any maximal torus in \( L \) has a stationary point on \( W^n \), thus \( H_x \) has maximal rank. Furthermore \( \pi_1(L/N_x) \cong N_x/H_x \), so that \( F_x \) has \((N_x/H_x)\) components. Since \( N_x \) must act transitively on the components of \( F_x \), it follows that \( B_x = H_x \). This implies that \( G_x = H_x \), so all the isotropy groups are connected and conjugate. Thus \((L, W^n)\) defines a compact fibration of \( W^n \) by \( L/H_x \) which must be proper, and this contradicts [1].

This is the extension of the Montgomery-Samelson theorem referred to in our title. It should be kept in mind that there is a conjecture to the effect that a compact connected Lie group acting on a sphere with one stationary point must have another. The Montgomery-Samelson theorem supports this conjecture and is a step toward a solution.

**References**


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1 The principal result of [1] asserts that an open simply connected manifold whose one-point compactification is again a manifold cannot be fibred by a compact fibre.