A NOTE ON STOCHASTIC APPROXIMATION

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1. A theorem on convergence of a sequence of random variables is proved in §2. In §3 this theorem is applied to prove convergence of a class of stochastic approximation procedures. The author was unable to verify whether the theorem of §3 could be derived from the general convergence theorem for stochastic approximation procedures due to Dvoretzky [1]. At any rate the method of proof used here appears to have some independent interest.

2. Let $\{X_n\}$ and $\{Y_n\}$ be infinite sequences of random variables. Let $I\{\cdots\}$ be the indicator (set characteristic function) of the set in brackets and for any random variable U let U^+ and U^- be the positive and negative part, respectively, of U. Then we have

THEOREM 1. *Suppose*

(2.1) $\quad X_n - \sum_{j=1}^{n} Y_j$ *converges with probability one* (*w.p.* 1).

(2.2) *For some positive integer k and every $\epsilon > 0$*

$$\sum_{n=k}^{\infty} I\{X_n \geqq \epsilon, \cdots, X_{n-k+1} \geqq \epsilon\} Y_n^{+} < +\infty \ w.p.\ 1$$

and

$$\sum_{n=k}^{\infty} I\{X_n \leqq -\epsilon, \cdots, X_{n-k+1} \leqq -\epsilon\} Y_n^{-} > -\infty \ \text{w.p. } 1.$$

Then X_n converges w.p. 1 *if and only if* $\lim_{n\to\infty} Y_n = 0$ *w.p.* 1.

PROOF. The necessity is immediate from (2.1). To prove sufficiency assume that we restrict ourselves to sample sequences in the set of probability one defined by the hypotheses of the theorem. Now if $\{X_n\}$ is any such sequence and $\lim_{n\to\infty} X_n = +\infty$ then, from (2.1), $\sum_{n=1}^{\infty} Y_n = +\infty$. But this clearly violates the first part of (2.2) for arbitrary $\epsilon > 0$. Similarly if $\lim_{n\to\infty} X_n = -\infty$. Hence $P\{\lim_{n\to\infty} X_n = +\infty\} = P\{\lim_{n\to\infty} X_n = -\infty\} = 0$. Now suppose $\{X_n\}$ is a sample sequence with $\liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n$. Assume $\limsup_{n\to\infty} X_n > 0$ and choose numbers a and b with $\liminf_{n\to\infty} X_n < a$ and $0 < a < b$

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$<\limsup_{n\to\infty} X_n$. Then it follows from the hypotheses that there are infinitely many pairs of positive integers (n_1, n_2) satisfying

(i) $n_1 < n_2$,

(ii) $X_n \geqq a/2, \cdots, X_{n-k+1} \geqq a/2$ for $n_1 \leqq n \leqq n_2$,

(iii) $X_{n_2} \geqq b$, $X_{n_1} \leqq a$.

Then $X_{n_2} - X_{n_1} \geqq b - a$ and it follows from (2.1) that $\sum_{j=n_1+1}^{n_2} Y_j \geqq (b-a)/2$ for n_1 sufficiently large. On the other hand if we apply the first part of (2.2) with $\epsilon = a/2$ we find that $\sum_{j=n_1+1}^{n_2} Y_j^+$ approaches zero as n_1 increases. Consequently we have a contradiction. A similar argument applies to those samples sequences for which $\limsup_{n\to\infty} X_n \leqq 0$, and the theorem is proved.

3. For each real number x let U_x be an integrable random variable with distribution function $H(u|x)$ and mean value $M(x)$. In many problems it is of interest to find a root of the equation $M(x) = 0$ assuming that the distribution functions $H(u|x)$ and the function $M(x)$ are unknown but that one may make observations on the random variable U_x for any desired value x. To this end Robbins and Monro [2] proposed a stochastic approximation scheme given by

$$x_{n+1} = x_n + a_n u_n \tag{3.0}$$

where x_1 is an arbitrary number, $\{a_n\}$ is a sequence of positive numbers and u_n is a random variable distributed according to $H(u|x_n)$. Robbins and Monro showed that under appropriate conditions x_n converges in probability to a root of the equation $M(x) = 0$. Much further work has since been done on this problem. For a description of this work and a list of references see Derman [3]. In this paper we shall consider a class of stochastic approximation procedures yielding estimates which converge with probability one to a root of the equation $M(x) = 0$.

Consider now the following set of conditions on the stochastic structure involved.

$$|M(x)| \leqq M < \infty. \tag{3.1}$$

$$\int_{-\infty}^{+\infty} [u - M(x)]^2 dH(u|x) \leqq \sigma^2 < \infty. \tag{3.2}$$

$$M(x) < 0 \quad \text{for} \quad x < \beta, \qquad M(x) > 0 \quad \text{for} \quad x > \beta. \tag{3.3}$$

(3.4) $\inf_{x\in I} |M(x)| > 0$ for every closed bounded interval I not containing β.

The generalization of the Robbins-Monro scheme we have in mind is motivated by the idea that one may not wish to take the nth observation from the distribution $H(u|x_n)$ but instead from the distribution $H(u|z_n)$ where z_n is a point determined in same fashion from a number of previous observations. To this end we consider for each positive integer n two measurable functions $f_n(x_1, \cdots, x_n)$ and $g_n(x_1, \cdots, x_n)$. Assume that the functions $f_n(x_1, \cdots, x_n)$ and $g_n(x_1, \cdots, x_n)$ satisfy: (3.5) There exists a positive integer k and for every real number r and every positive number η a positive integer $n(r, \eta)$ such that $n \geqq n(r, \eta)$, $x_n \geqq r+\eta, \cdots, x_{n-k+1} \geqq r+\eta$ implies $f_n(x_1, \cdots, x_n) > r$ and $n \geqq n(r, \eta)$, $x_n \leqq r-\eta, \cdots, x_{n-k+1} \leqq r-\eta$ implies $f_n(x_1, \cdots, x_n) < r$.

(3.6) $\sum_{n=1}^{\infty} |g_n(x_1, \cdots, x_n)| < \infty$ for every sequence $\{x_n\}$ of real numbers.

Let $\{a_n\}$ be a sequence of positive numbers satisfying

$$\sum_n a_n = \infty, \qquad \sum_n a_n^2 < \infty. \tag{3.7}$$

Let x_1 be an arbitrary number. Define x_{n+1} recursively by the equation

$$x_{n+1} - x_n = g_n(x_1, \cdots, x_n) - a_n u_n \tag{3.8}$$

where u_n is an observation on the random variable distributed according to $H(u|f_n(x_1, \cdots, x_n))$. Then we have

THEOREM 2. *If $\{x_n\}$ is the process* (3.8) *and if conditions* (3.1) *through* (3.7) *hold then* $P\{\lim_{n\to\infty} x_n = \beta\} = 1$.

PROOF. We shall assume without loss of generality that $\beta = 0$. We prove the theorem by first showing that Theorem 1 applies and then verifying that the resulting limit is the correct one. To do this we make the following identifications with the random variables of Theorem 1: Let $X_0 = 0$, $Y_1 = 0$, $X_n = x_n$, and $Y_n = g_{n-1}(x_1, \cdots, x_{n-1}) - a_{n-1}M(f_{n-1})$. Then

$$X_n - \sum_{j=1}^{n} Y_j = \sum_{j=1}^{n} [(X_j - X_{j-1}) - Y_j] = \sum_{j=1}^{n-1} a_j[M(f_j) - u_j].$$

Now from (3.2) and (3.7) we have that $\sum_n a_n^2 E\{[M(f_j) - u_j]^2\} \leqq \sigma^2 \sum_n a_n^2 < \infty$ and it follows from straight forward considerations (see e.g. Loeve [2, p. 387]) that (2.1) holds. Now if $x_n \geqq \epsilon, \cdots, x_{n-k+1} \geqq \epsilon$ then we know from (3.5) that for n sufficiently large $f_n(x_1, \cdots, x_n) > 0$. This together with (3.3) and (3.6) establishes the first part of (2.2) and a similar argument takes care of the second part. Now

$$\begin{aligned} |Y_n| &= |g_{n-1}(x_1, \cdots, x_{n-1}) - a_{n-1}M(f_{n-1})| \\ &\leqq |g_{n-1}(x_1, \cdots, x_{n-1})| + a_{n-1}M \end{aligned}$$

and from (3.1), (3.6), and (3.7) we see that $\lim_{n\to\infty} Y_n = 0$ w.p. 1. Hence Theorem 1 applies. Now suppose $P\{\lim_{n\to\infty} x_n = 0\} < 1$. Then there exist numbers a and b with (say) $0 < a < b$ such that $P\{a \leqq \lim_{n\to\infty} x_n \leqq b\} > 0$. Then from (3.5) it follows that for any such sample sequence $\{x_n\}$ we have $0 < a \leqq \lim_{n\to\infty} f_n(x_1, \cdots, x_n) = \lim_{n\to\infty} x_n \leqq b$. It follows from (3.3), (3.4), and (3.6) that

$$\sum_{n=1}^{\infty} [g_n(x_1, \cdots, x_n) - a_n M(f_n)] = -\infty.$$

But then $\sum_{n=1}^{\infty} a_n[M(f_n) - u_n] = +\infty$ and this happens with probability zero. Consequently $P\{\lim_{n\to\infty} x_n = 0\} = 1$.

We mention in passing that this same type of modification may be applied to the stochastic approximation procedure discussed by Kiefer and Wolfowitz [5] which estimates the point where the regression function achieves a maximum.

References

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