THE GROUP OF ORDER PRESERVING AUTOMORPHISMS OF AN ORDERED ABELIAN GROUP

PAUL CONRAD[1]

In this note we shall use the terminology and notation from [2, pp. 516–517]. In particular, order will always mean linear order. Some answers are given to the question: When can the group $\mathfrak{A}$ of o-automorphisms of an abelian o-group G be ordered? We prove that $\mathfrak{A}$ can be ordered provided that the rank of G is well ordered and the o-automorphism group of each component of G is isomorphic to a subgroup of the positive rationals. Thus every torsion free abelian group G admits an ordering for which the corresponding $\mathfrak{A}$ can be ordered. Our results provide examples of nonabelian o-groups, and they can be used to obtain information about the underlying group G.

1. **Representation of an automorphism group by a group of matrices.** Whenever possible we shall represent automorphisms by matrices. This is a straightforward generalization of the classical case (for example, see Kurosh [3, p. 156]), and no proofs will be given. Let Γ be an ordered set, and for each $\gamma \in \Gamma$, let D_γ be an abelian group. We use the same symbol 0 to denote the identity element of each of the D_γ. Let E_γ be the ring of endomorphisms of D_γ, and let $H(D_\alpha, D_\beta)$, where $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$, be the group of homomorphisms of D_α into D_β. Finally let G be the restricted direct sum of the D_γ, and let Δ be the set of all square matrices $[\pi_{\alpha\beta}]$, where $\alpha, \beta \in \Gamma$, $\pi_{\alpha\alpha} \in E_\alpha$, and $\pi_{\alpha\beta} \in H(D_\alpha, D_\beta)$ for $\alpha \neq \beta$, that satisfy:

(*) For α fixed and each $d \in D_\alpha$, $d\pi_{\alpha\beta} \neq 0$ for at most a finite number of the β.

If $(\cdots, d_\alpha, \cdots) \in G$ and $[\pi_{\alpha\beta}] \in \Delta$, then

$$(\cdots, d_\alpha, \cdots)[\pi_{\alpha\beta}] = (\cdots, \textstyle\sum_{\alpha\in\Gamma} d_\alpha \pi_{\alpha\beta}, \cdots) \in G.$$

It is easy to verify that this mapping is an endomorphism of G. Conversely if π is an endomorphism of G, then $(0, \cdots, 0, a_\alpha, 0, \cdots, 0)\pi = (\cdots, b_\beta, \cdots)$ where at most a finite number of the b_β are nonzero. Define $a_\alpha \pi_{\alpha\beta} = b_\beta$, then we have a mapping of π upon $[\pi_{\alpha\beta}] \in \Delta$. This mapping is an isomorphism of the ring of all endomorphisms of G onto Δ. Suppose that each D_γ is an o-group, and define $(\cdots, d_\gamma, \cdots) \in G$ positive if it is not zero and the nonzero com-

Received by the editors November 9, 1957.

[1] This research was supported by a grant from the National Science Foundation.

ponent d_γ with greatest subscript γ is positive. Then G is an abelian o-group, and for each $\gamma \in \Gamma$, $C_\gamma = \{(\cdots, g_\alpha, \cdots) \in G: g_\alpha = 0$ for all $\alpha > \gamma\}$ is a convex subgroup of G. Let $\mathfrak{A}$ be the group of all o-automorphisms of G, and for each $\gamma \in \Gamma$, let $\mathfrak{D}_\gamma$ be the group of all o-automorphisms of D_γ. Let $T = T(\Gamma, D_\gamma)$ be the multiplicative semigroup of all the triangular matrices $[\pi_{\alpha\beta}] \in \Delta$ that satisfy:

(a) $\pi_{\gamma\gamma} \in \mathfrak{D}_\gamma$ for all $\gamma \in \Gamma$.

(b) If $\alpha, \beta \in \Gamma$, $\beta > \alpha$ and $d \in D_\alpha$, then $d\pi_{\alpha\beta} = 0$.

$[\pi_{\alpha\beta}] \in T$ corresponds to an o-isomorphism of G into itself, and $[\pi_{\alpha\beta}]$ corresponds to an o-automorphism of G if and only if $[\pi_{\alpha\beta}]$ is a unit in T. Let $U = U(\Gamma, D_\gamma)$ be the group of units of T.

LEMMA 1. *If each of the convex subgroups C_γ of G is invariant with respect to $\mathfrak{A}$, then $\mathfrak{A}$ is isomorphic to U.*

PROOF. By the isomorphism defined in the preceding discussion, $\mathfrak{A}$ is mapped onto a subgroup of the group of all matrices in Δ that have inverses in Δ. Let $\pi \in \mathfrak{A}$ and let $[\pi_{\alpha\beta}]$ be the corresponding matrix. For each $\gamma \in \Gamma$, $C_\gamma[\pi_{\alpha\beta}] = C_\gamma$. It follows that $[\pi_{\alpha\beta}]$ satisfies (b). Then since π is an o-automorphism, $[\pi_{\alpha\beta}]$ satisfies (a).

COROLLARY. *If each D_γ is a subgroup of the reals (naturally ordered), and the only o-permutation of Γ is the identity permutation, then $\mathfrak{A}$ is isomorphic to U.*

For under these restrictions it is easy to verify that *every* convex subgroup of G is invariant with respect to $\mathfrak{A}$.

REMARK. If each D_γ is the group of reals, then $\mathfrak{A}$ is isomorphic to U if and only if the only o-permutation of Γ is the identity permutation.

A subgroup S of G is a *c-subgroup* if for every $\gamma \in \Gamma$ and $d \in D_\gamma$ there exists an element $(\cdots, g_\alpha, \cdots) \in S$ such that $g_\gamma = d$ and $g_\alpha = 0$ for all $\alpha > \gamma$.

LEMMA 2. *Suppose that each D_γ is a nonzero d-closed subgroup of the reals. Then $U = T$ if and only if Γ is well ordered.*

PROOF. Each $[\pi_{\alpha\beta}] \in T$ induces an o-isomorphism of G onto a c-subgroup of G. If Γ is well ordered, then G has no proper c-subgroups [1, p. 8]. Thus $G[\pi_{\alpha\beta}] = G$ for all $[\pi_{\alpha\beta}] \in T$, hence $U = T$.

Conversely assume that Γ is not well ordered. Then we can find a set $\{h_{\gamma_i}\}_{i=1}^\infty$ of nonzero elements such that $h_{\gamma_i} \in D_{\gamma_i}$ and $\gamma_1 > \gamma_2 > \gamma_3 > \cdots$. Let $h_{\gamma_i}^*$ be the smallest d-closed subgroup of D_{γ_i} that contains h_{γ_i}, and let B be the restricted direct sum of the $h_{\gamma_i}^*$. Define $h_{\gamma_i}\pi = h_{\gamma_{i+1}} + h_{\gamma_i}$. Since B is a vector space over the rationals and the h_{γ_i} form a basis for B, there is a unique extension of π to an o-isomorphism σ of B into B. Note that $(0, \cdots, 0, h_{\gamma_1}, 0, \cdots, 0)$ does not belong to $B\sigma$. For each γ_i let K_{γ_i} be a subgroup of D_{γ_i} such that $D_{\gamma_i} = h^*_{\gamma_i} \oplus K_{\gamma_i}$. This is possible since the $h^*_{\gamma_i}$ are d-closed. Let E be the restricted direct sum of the K_{γ_i} for $i = 1, 2, \cdots$ and the D_α for all $\alpha \neq \gamma_1, \gamma_2, \cdots$. Then $G = B \oplus E$. For each $g = b + e$ in G let $g\tau = b\sigma + e$. It is easy to verify that τ is an o-isomorphism of G onto a proper subgroup of G, and that the corresponding matrix $[\pi_{\alpha\beta}]$ belongs to[2] T. Thus $U \neq T$.

COROLLARY. *If each D_γ is a subgroup of the reals and Γ is well ordered, then $\mathfrak{A}$ is isomorphic to U and $U = T$.*

2. **The group $\mathfrak{B}$.** For the rest of this paper G will denote an abelian o-group, and Γ will denote the set of all pairs of convex subgroups G^γ, G_γ of G such that G^γ covers G_γ.

LEMMA 3. *If G^* is the d-closure of G and π is an o-automorphism of G, then there exists a unique o-automorphism π^* of G^* such that $g\pi^* = g\pi$ for all $g \in G$. Moreover the mapping $\pi \rightarrow \pi^*$ is an isomorphism of the group $\mathfrak{A}$ of all o-automorphisms of G into the group $\mathfrak{A}^*$ of o-automorphisms of G^*. Thus $\mathfrak{A}$ can be ordered if $\mathfrak{A}^*$ can be ordered.*

PROOF. For each $g^* \in G^*$ there exists a positive integer n such that $ng^* \in G$. Let $g^*\pi^*$ be the solution of $nx = (ng^*)\pi$. The verification that $\pi^* \in \mathfrak{A}^*$ and that the mapping $\pi \rightarrow \pi^*$ is an isomorphism is not difficult.

For the remainder of this section we shall assume that G is d-closed. Let $\mathfrak{B} = \{\pi \in \mathfrak{A}: (G_\gamma + g)\pi = G_\gamma + g$ for all $\gamma \in \Gamma$ and all $g \in G^\gamma\}$. That is, $\mathfrak{B}$ consists of all those o-automorphisms of G that induce the identity automorpism on each of the components G^γ/G_γ of G. $\mathfrak{B}$ is a normal subgroup of $\mathfrak{A}$. For let $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$ and $a \in G^\gamma$. $G^\gamma\alpha$ and $G_\gamma\alpha$ are convex subgroups and $G^\gamma\alpha$ covers $G_\gamma\alpha$. Therefore $(G_\gamma + a)\alpha\beta\alpha^{-1} = (G_\gamma\alpha + a\alpha)\beta\alpha^{-1} = (G_\gamma\alpha + a\alpha)\alpha^{-1} = G_\gamma + a$. Since G is d-closed, G^γ and G_γ are d-closed for all $\gamma \in \Gamma$. Thus $G^\gamma = G_\gamma \oplus D_\gamma$, where D_γ is a d-closed subgroup of G. Therefore D_γ is o-isomorphic to a subgroup of the reals. Let $\mathfrak{B}^\gamma(\mathfrak{B}_\gamma)$ denote the group of all o-automorphisms of $G^\gamma(G_\gamma)$ that induce the identity automorphism on all components. By Lemma 1, $\mathfrak{B}^\gamma$ is isomorphic to the multiplicative group of all matrices of the form

$$[a] = \begin{bmatrix} a_{11} & \theta \\ a_{21} & 1 \end{bmatrix}$$

[2] The author originally asserted that this was true for an arbitrary choice of the complementary direct summand of B, but the referee killed this with a nice counter example.

where $a_{11} \in \mathfrak{B}_\gamma$ and $a_{21} \in H(D_\gamma, G_\gamma)$. Here and for the rest of this note 1 will denote an identity automorphism and θ a zero homomorphism. It will be clear what group 1 refers to (D_γ in this case), and what group θ refers to ($H(G_\gamma, D_\gamma)$ in this case). In order to show that an ordering of $\mathfrak{B}_\gamma$ can be extended to an ordering of $\mathfrak{B}^\gamma$ we need the following lemma.

LEMMA 4. *If A is an abelian o-group, then $H(G, A)$ can be ordered so that if $h \in H$ is positive and α is an o-automorphism of A, then $h\alpha$ is positive.*

PROOF. Since G is torsion free and d-closed, it is a vector space over the rationals $\mathfrak{R}$. Choose and well order a basis $g_1, g_2, \cdots$ for G. Define $h \in H$ positive if $h \neq \theta$ and $g_i h > 0$ in A, where g_i is the first basis element for which $g_i h \neq 0$.

COROLLARY. *$\mathfrak{B}_\gamma$ is isomorphic to a subgroup of $\mathfrak{B}^\gamma$ and any ordering of $\mathfrak{B}_\gamma$ can be extended to an ordering of $\mathfrak{B}^\gamma$.*

PROOF. Order $H(D_\gamma, G_\gamma)$ so that if $h \in H$ is positive and $\pi \in \mathfrak{B}_\gamma'$ then $h\pi$ is positive. Define $[a]$ positive if $a_{11} > 1$ or $a_{11} = 1$ and $a_{21} > \theta$. It is easy to verify that this definition orders $\mathfrak{B}^\gamma$.

THEOREM 1. *If Γ is well ordered, then $\mathfrak{B}$ can be ordered.*

PROOF. G is isomorphic to the restricted direct sum of the d-closed subgroups D_γ of the reals, where $D_\gamma \cong G^\gamma / G_\gamma$ [1, pp. 19 and 8]. Thus by the corollary to Lemma 2, $\mathfrak{A}$ is isomorphic to the group of matrices $T(\Gamma, D_\gamma)$. Clearly $[\pi_{\alpha\beta}] \in T$ corresponds to an element of $\mathfrak{B}$ if and only if $\pi_{\gamma\gamma} = 1$ for all $\gamma \in \Gamma$. $0 = G_1 \subset G^1 = G_2 \subset G^2 = \cdots \subset G_\omega \subset G^\omega = \cdots$. Thus for each $\gamma \in \Gamma$, $\mathfrak{B}^\gamma$ is isomorphic to the group of all matrices of the form

$$\begin{bmatrix} 1 & & & & & & \\ a_{21} & 1 & & \theta & & & \\ \cdot & & \ddots & & & & \\ \cdot & & & \ddots & & & \\ a_{\gamma 1} & \cdots & & 1 & & & \\ & & & & \ddots & & \\ & \theta & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Therefore without loss of generality $1 = \mathfrak{B}_1 \subset \mathfrak{B}^1 = \mathfrak{B}_2 \subset \mathfrak{B}^2 = \cdots \subset \mathfrak{B}_\omega \subset \mathfrak{B}^\omega = \cdots$. If $\gamma \in \Gamma$ has no immediate predecessor, then $\mathfrak{B}_\gamma = \bigcup_{\alpha < \gamma} \mathfrak{B}^\alpha$. Also $\mathfrak{B} = \bigcup_{\gamma \in \Gamma} \mathfrak{B}^\gamma$. It follows at once from the corollary to Lemma 4 that $\mathfrak{B}$ can be ordered.

Note that the theorem remains true without the restriction that G is d-closed. For if G has well ordered rank (that is, Γ is well ordered), then so does the d-closure of G.

COROLLARY I. *If A is an abelian o-group with well ordered rank, and $\mathfrak{A}^*$ is the group of all o-automorphisms of A that induce the identity automorphism on each of the components of A, then $\mathfrak{A}^*$ can be ordered.*

Let π be a value preserving o-isomorphism of G into G. That is, $g \in G^\gamma \setminus G_\gamma$ if and only if $g\pi \in G^\gamma \setminus G_\gamma$. Define $(G_\gamma + g)\pi' = G_\gamma + g\pi$ for all $g \in G_\gamma$. Then π' is an o-isomorphism of G^γ/G_γ into G^γ/G_γ. $T(\Gamma, D_\gamma)$ is isomorphic to the semigroup of all value-preserving o-isomorphisms of G into G that induce an o-automorphism on each of the components G^γ/G_γ of G. In the proof of the theorem it was shown that $T = U$.

COROLLARY II. *If Γ is well ordered, then every value preserving o-isomorphism of G into G that induces an o-automorphism on each of the components of G is an o-automorphism of G.*

REMARK. Corresponding to Lemma 2 we have the following: *Every value preserving o-isomorphism of G into G is an o-automorphism if and only if Γ is well ordered and no G^γ/G_γ admits an o-isomorphism onto a proper subgroup of itself.* It is not necessary to assume that G is d-closed, since both conditions imply that G is d-closed. The proof is a consequence of Lemma 2, and the embedding theorem [**1**, p. 19].

If Γ is well ordered, then by the corollary to Lemma 2, $\mathfrak{A}/\mathfrak{B}$ is isomorphic to the group of all diagonal matrices in $T(\Gamma, D_\gamma)$. The o-automorphism group of the subgroup D_γ of the reals is isomorphic to a subgroup of the positive reals, hence it has a natural order. Therefore $\mathfrak{A}/\mathfrak{B}$ can be ordered. Thus an ordering of $\mathfrak{B}$ can be extended to an ordering of $\mathfrak{A}$ so that $\mathfrak{B}$ is a convex subgroup if and only if $\alpha^{-1}\beta\alpha$ is positive for $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{B}$ whenever β is positive in $\mathfrak{B}$ (Levi [**4**]). In §4 we show that $\mathfrak{B}$ can be so ordered, provided that the components of G admit only rational o-automorphisms. In §3 we order $\mathfrak{A}$ by the method we used to order $\mathfrak{B}$.

3. **The group $\mathfrak{A}$.** *Throughout this section assume that G is a d-closed abelian o-group with well ordered rank.* Let $\mathfrak{A}_\gamma(\mathfrak{A}^\gamma)$ be the group of all o-automorphisms of $G_\gamma(G^\gamma)$. $G^\gamma = G_\gamma \oplus D_\gamma$. Thus $\mathfrak{A}^\gamma$ is isomorphic to the multiplicative group $\mathfrak{M}$ of all matrices of the form

$$[a] = \begin{bmatrix} a_{11} & \theta \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_{11} \in \mathfrak{A}_\gamma$, $a_{21} \in H(D_\gamma, G_\gamma)$ and a_{22} is an o-automorphism of D_γ.

In particular, $\mathfrak{A}_\gamma$ is isomorphic to a subgroup of $\mathfrak{A}^\gamma$. Suppose that $\mathfrak{A}_\gamma$ and H are ordered and define $[a]$ positive if $a_{22}>1$ or $a_{22}=1$ and $a_{11}>1$ or $a_{22}=1$, $a_{11}=1$ and $a_{21}>\theta$. Then it is easy to verify that for any nonzero matrix a in $\mathfrak{M}$, either a is positive or a^{-1} is positive. Also if a and b are positive matrices in $\mathfrak{M}$, then ab is positive. Finally suppose that $a, b\in\mathfrak{M}$ and b is positive.

$$a^{-1}ba = \begin{bmatrix} \overset{-1}{a_{11}} b_{11}a_{11} & \theta \\ \overset{-1}{a_{22}} (b_{21}a_{11} + b_{22}a_{21} - a_{21}\overset{-1}{a_{11}} b_{11}a_{11}) & b_{22} \end{bmatrix}.$$

If $b_{22}>1$ or $b_{22}=1$ and $b_{11}>1$, then $a^{-1}ba$ is positive. Suppose that $b_{22}=1$, $b_{11}=1$ and $b_{21}>\theta$. Then

$$a^{-1}ba = \begin{bmatrix} 1 & \theta \\ \overset{-1}{a_{22}} b_{21}a_{11} & 1 \end{bmatrix}.$$

Thus if we wish to extend the order of $\mathfrak{A}_\gamma$ to an order of $\mathfrak{A}^\gamma$ we must order $H(D_\gamma, G_\gamma)$ so that $a_{22}^{-1}b_{21}a_{11}$ is positive. Now without loss of generality, D_γ is a d-closed subgroup of the reals. Hence the only o-automorphisms of D are multiplications by some positive real numbers. In particular, a multiplication by a positive rational is an o-automorphism, since D is d-closed.

Lemma 5. *If the only o-automorphisms of D_γ are multiplications by positive rationals, then $H(D_\gamma, G_\gamma)$ can be ordered so that if $h\in H$ is positive, then $\alpha h\beta$ is positive for all o-automorphisms α and β of D_γ and G_γ respectively.*

Proof. Choose and well order a basis $d_1, d_2, \cdots$ for D. Define $h\in H$ positive if $h\neq\theta$ and $d_ih>0$, where d_i is the first basis element for which $d_ih\neq 0$. $d_j\alpha h\beta=(rd_j)h\beta=(r(d_jh))\beta$, where r is a positive rational and α and β are o-automorphisms of D_γ and G_γ. Thus $d_j\alpha h\beta$ is positive if and only if d_jh is positive.

Theorem 2. *If G is an abelian d-closed o-group with well ordered rank and each component G^γ/G_γ has its group of o-automorphisms isomorphic to the positive rationals, then $\mathfrak{A}$ can be ordered so that $\mathfrak{B}$ is a convex subgroup.*

The proof is entirely similar to the proof of Theorem 1.

Corollary. *Any torsion free abelian group can be ordered in such a way that the resulting group of o-automorphisms can also be ordered.*

Proof. Let A be a torsion free abelian group. Choose and well order a basis $a_1, a_2, \cdots$ for the d-closure A^* of A. Then A^*

$=\Re a_1 \oplus \Re a_2 \oplus \cdots$. Define $a = \cdots + r_i a_i + \cdots$ positive if $a \neq 0$ and the nonzero coefficient r_i with greatest subscript is positive. Then A^* satisfies the hypotheses of the theorem. Hence the group of o-automorphisms $\mathfrak{A}^*$ of A^* can be ordered, and the group $\mathfrak{A}$ of o-automorphisms of A is isomorphic to a subgroup of $\mathfrak{A}^*$.

REMARK. Note that $\mathfrak{A}^*$ is isomorphic to the group K of all row finite triangular matrices $[\pi_{\alpha\beta}]$, where $\alpha, \beta \in \Gamma$ the rank of A^*, $\pi_{\alpha\alpha}$ is a positive rational, $\pi_{\alpha\beta}$ is a rational, and $\pi_{\alpha\beta} = 0$ for $\alpha < \beta$. Then K can be ordered. In fact, the following definition orders K. $[\pi_{\alpha\beta}]$ is positive if (a) there exists an $\alpha \in \Gamma$ such that $\pi_{\alpha\alpha} > 1$ and $\pi_{\beta\beta} = 1$ for all $\beta < \alpha$, or (b) $\pi_{\alpha\alpha} = 1$ for all $\alpha \in \Gamma$ and there exist $\alpha, \beta \in \Gamma$ such that $\pi_{\alpha\beta} > \theta$, all elements above $\pi_{\alpha\beta}$ in the diagonal containing $\pi_{\alpha\beta}$ and parallel to the main diagonal are zero, and all elements in the diagonals parallel to but distinct from the main diagonal and above the diagonal containing $\pi_{\alpha\beta}$ are zero.

4. **A generalization of Theorem** 2. Our proof of Theorem 2 does not permit the dropping of the hypothesis that G is d-closed. For example, if one of the components of G is isomorphic to $\mathfrak{J} \oplus \mathfrak{J} 2^{1/2}$, where $\mathfrak{J}$ is the group of integers, then the corresponding components of the d-closure of G is isomorphic to $\Re \oplus \Re 2^{1/2}$. The o-automorphism group of $\Re \oplus \Re 2^{1/2}$ is isomorphic to the group of all positive elements of $\Re \oplus \Re 2^{1/2}$.

THEOREM 3. *If G is an abelian o-group with well ordered rank, and each component G^γ / G_γ has its group of o-automorphisms isomorphic to a subgroup of the positive rationals, then $\mathfrak{A}$ can be ordered.*

PROOF. Well order $\prec$ the elements of G so that if $a, b \in G$ and $V(a) < V(b)$, then $a \prec b$, where $V(a)$ is the value of a with respect to the given order of G. That is,

$$\underbrace{0 \prec g_{11} \prec g_{12} \prec \cdots}_{G^1 \setminus G_1} \quad \underbrace{g_{21} \prec g_{22} \prec \cdots}_{G^2 \setminus G_2} \quad \cdots \underbrace{g_{\omega 1} \prec g_{\omega 2} \prec \cdots}_{G^\omega \setminus G_\omega} \cdots .$$

For each $\pi \neq 1$ in $\mathfrak{A}$ there exists a least element g in the well ordering $\prec$ such that $g\pi \neq g$. Denote this least element by $L(\pi)$, and define π positive if $L(\pi)\pi > L(\pi)$. If $\pi \neq 1$ and π is not positive, then $L(\pi^{-1}) = L(\pi)$ and $L(\pi)\pi < L(\pi)$. Thus $L(\pi^{-1}) = L(\pi) < L(\pi)\pi^{-1} = L(\pi^{-1})\pi^{-1}$, hence π^{-1} is positive. Let α and β be positive elements of $\mathfrak{A}$. If $L(\alpha) = L(\beta)$ or $L(\alpha) \prec L(\beta)$ and $g \prec L(\alpha)$, then $g\alpha\beta = g\beta = g$ and $L(\alpha)\alpha\beta > L(\alpha)\beta \geqq L(\alpha)$. If $L(\beta) \prec L(\alpha)$ and $g \prec L(\beta)$, then $g\alpha\beta = g\beta = g$ and $L(\beta)\alpha\beta = L(\beta)\beta > L(\beta)$. Therefore $\alpha\beta$ is positive.

Finally suppose that $\alpha, \beta \in \mathfrak{A}$ and that β is positive. Let $g \in G^\gamma \setminus G_\gamma$.

Then $g\alpha \equiv (m/n)g$ mod G_γ, hence $n(g\alpha) = mg + d$ where m and n are positive integers and $d \in G_\gamma$. In particular, $d \dashv g$. If $g \dashv L(\beta)$, then $n(g\alpha\beta\alpha^{-1}) = (mg+d)\beta\alpha^{-1} = (mg+d)\alpha^{-1} = ng$, hence $g\alpha\beta\alpha^{-1} = g$. If $g = L(\beta)$, then $n(L(\beta)\alpha\beta\alpha^{-1}) = (mL(\beta)+d)\beta\alpha^{-1} = (mL(\beta)\beta + d)\alpha^{-1} > (mL(\beta)+d)\alpha^{-1} = nL(\beta)$, hence $L(\beta)\alpha\beta\alpha^{-1} > L(\beta)$. Therefore $\alpha\beta\alpha^{-1}$ is positive and $\mathfrak{A}$ is ordered.

Note that for this ordering of $\mathfrak{A}$, $\mathfrak{B}$ is not a convex subgroup. However, the above definition does order $\mathfrak{B}$ for any abelian o-group with well ordered rank. Thus we have a second proof of Theorem 1. Moreover, this ordering of $\mathfrak{B}$ can be extended to an ordering of $\mathfrak{A}$ provided that each component of G has its group of o-automorphisms isomorphic to a subgroup of the positive rationals. For if $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$ and β is positive, then the above proof shows that $\alpha\beta\alpha^{-1}$ is positive. Thus we have a proof of Theorem 2 that does not use the hypothesis that G is d-closed.

Bibliography

1. P. Conrad, *Embedding theorems for abelian groups with valuations*, Amer. J. Math. vol. 75 (1953) pp. 1–29.

2. ———, *Extensions of ordered groups*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 516–528.

3. A. G. Kurosh, *The theory of groups*, Chelsea, New York.

4. F. W. Levi, *Ordered groups*, Proceedings of the Indian Academy of Sciences, vol. 16 (1942) pp. 256–263.

Tulane University