$$\frac{k\lambda - (k-1)}{(k-1)\lambda - (k-2)} = \frac{k'\lambda - (k'-1)}{(k'-1)\lambda - (k'-2)}$$

the net result of which is $(k-k')(\lambda-1)^2=0$. Since $0 \leqq k < k' \leqq n$, and the characteristic p of F is larger than n, $k-k' \not\equiv 0(p)$ and so $(\lambda-1)^2 = 0$ results. Hence $\lambda=1$ and the theorem is proved.

We point out that the argument used above also works, up to a point in a Banach algebra. The conclusion one can reach is that the spectrum of $ab^{-1}a^{-1}b$ is invariant under the transformation $(2\lambda-1)/\lambda$.

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SOME ORTHOGONAL FUNCTIONS CONNECTED WITH POLYNOMIAL IDENTITIES

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If $P(x)$ is an arbitrary polynomial of sufficiently small degree then there are various ways of choosing an integer N and coefficients c_j such that

$$\sum_{j=0}^{N} c_j P(x+j) = 0 \tag{1}$$

for all x. In each of [**3**; **4**] such an identity is proved. The present paper is devoted to a discussion of a connection between the coefficients in these identities and certain classes of orthonormal functions.

1. A polynomial identity and the Walsh functions. In [4] we proved the identity

$$\sum_{n=0}^{b_1^{k+1}-1} \omega^{v_b(n)} P(x+n) = 0, \tag{2}$$

where b and k are positive integers, $v_b(n)$ is the sum of the coefficients

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of n in base b, ω is an arbitrary bth root of unity other than 1, and $P(x)$ is any polynomial of degree $\leqq k$. For uniformity in notation with [1] we take $\omega = e^{2\pi i/b}$.

We now define, for each k, a function of x of period 1 whose values are precisely these coefficients on the successive intervals of $[0, 1)$ which are of length $1/b^{k+1}$.

DEFINITION 1.

$$\left.\begin{aligned} w_n(x) &= \omega^{v_b([b^{n+1}x])}, \quad 0 \leqq x < 1; \\ w_n(x+1) &= w_n(x) \text{ for all } x \end{aligned}\right\} \qquad n \geqq 0.$$

Note that these functions depend on b. We shall not make this explicit in our notation as b is to remain fixed.

Our first theorem states that $w_n(x)$ is the product of the first $n+1$ Rademacher functions of order b (see [1]). The Rademacher functions of order b are defined as follows.

$$\begin{aligned} r_0(x) &= \omega^{[bx]}, & 0 \leqq x \leqq 1; \\ r_n(x+1) &= r_n(x) = r_0(b^n x), & n \geqq 0. \end{aligned}$$

THEOREM 1.

$$w_n(x) = \prod_{j=0}^{n} r_j(x).$$

PROOF. Suppose first that $0 \leqq x \leqq 1$. Then the coefficient of b^j in the base b expansion of $[b^{n+1}x]$ is congruent modulo b to

$$[[b^{n+1}x]/b^j] = [b^{n-j+1}x].$$

Hence

$$v_b([b^{n+1}x]) \equiv [b^{n+1}x] + [b^n x] + \cdots + [bx] \pmod{b}.$$

Therefore

$$\begin{aligned} \prod_{j=0}^{n} r_j(x) &= r_0(x) r_0(bx) \cdot \cdots \cdot r_0(b^n x) \\ &= r_0(x) r_0(bx - [bx]) \cdots r_0(b^n x - [b^n x]) \\ &= \omega^{[bx]} \omega^{[b^2x - b[bx]]} \cdots \omega^{[b^{n+1}x - b[b^n x]]} \\ &= \omega^{[bx]} \omega^{[b^2 x]} \cdots \omega^{[b^{n+1}x]} = \omega^{v_b([b^{n+1}x])} = w_n(x). \end{aligned}$$

The theorem follows from periodicity.

The Walsh functions of order b (see [1]) are defined by

$$\begin{aligned} \psi_0(x) &= 1; \\ \psi_n(x) &= r_{n_1}^{a_1}(x) \cdots r_{n_k}^{a_k}(x) \end{aligned}$$

where the base b expansion of n is $a_1b^{n_1}+\cdots+a_kb^{n_k}$. Hence from Theorem 1 we conclude that the sequence $\{w_n(x)\}$ is an orthonormal set since each $w_n(x)$ is a Walsh function. We also note that $\{w_n(x)\}$ is a lacunary subsequence of $\{\psi_n(x)\}$.

2. **A similar situation with another polynomial identity.** In [3] we proved the identity

$$\sum_{n=0}^{b^k-1} u_b(n+1)P(x+n) = 0 \tag{3}$$

where $P(x)$ is a polynomial of degree less than $k(b-1)$ and the coefficients $u_b(n+1)$ are given in (4) below.

$b_j(n)$ is the coefficient of b^j in the base b expansion of n;

$$\begin{aligned} v_b(n) &= \sum_{j=0}^{\infty} b_j(n); \\ u_b(n+1) &= (-1)^{v_b(n)} \prod_{j=0}^{\infty} \binom{b-1}{b_j(n)}. \end{aligned} \tag{4}$$

An easy consequence of (4) (see [3]) is the identity

$$u_b(n+mb^k) = u_b(n)u_b(m+1) \text{ for } 1 \leqq n \leqq b^k, \qquad m \geqq 0. \tag{5}$$

Using the coefficients of (3) we construct a sequence of functions in analogy with the $w_n(x)$.

DEFINITION 2.

$$\begin{aligned} t_n(x) &= u_b(1+[b^{n+1}x]), \quad 0 \leqq x < 1; \\ t_n(x+1) &= t_n(x) \text{ for all } x \qquad\qquad n \geqq 0. \end{aligned}$$

Note that when $b=2$ we have $t_n(x)=w_n(x)$. This case is discussed in [3].

The natural analogues of the $r_n(x)$ out of which one can build the $t_n(x)$ would seem to be the functions $s_n(x)$ defined next.

DEFINITION 3.

$$s_0(x) = u_b(1+[bx]) = (-1)^{[bx]}\binom{b-1}{[bx]}, \qquad 0 \leqq x < 1;$$

$$s_n(x+1) = s_n(x) = s_0(b^n x), \qquad n \geqq 0.$$

THEOREM 2.

$$t_n(x) = \prod_{j=0}^{n} s_j(x).$$

PROOF. For $n=0$ this is evident. Hence we suppose it is true for

$n \leqq k$. Then for $0 \leqq x < 1$, $r = b^{k+1}x - [b^{k+1}x]$ we have

$$\prod_{j=0}^{k+1} s_j(x) = t_k(x)s_{k+1}(x) = u_b(1 + [b^{k+1}x])s_0(b^{k+1}x)$$
$$= u_b(1 + [b^{k+1}x])s_0(r) = u_b\,(1 + [b^{k+1}x])u_b(1 + [br])$$
$$\overset{*}{=} u_b(1 + [br] + b[b^{k+1}x]) = u_b(1 + [b^{k+2}x]) = t_{n+1}(x).$$

At the asterisk we used (5). The conclusion now follows from periodicity.

3. **Further properties of the** $s_n(x)$ **and** $t_n(x)$ **functions.** In this section we prove a theorem concerning the orthogonality of power products of the $s_n(x)$. This theorem includes the orthogonality of the $t_n(x)$. We first prove two lemmas.

LEMMA 1. $s_0(x) = u_b(k - [(k-1)/b])$ *for* $(k-1)/b \leqq x < k/b$.

PROOF. Write $k - 1 = [(k-1)/b]b + r$, $0 \leqq r < b$. Then $x = [(k-1)/b] + (r+s)/b$, $0 \leqq s < 1$. Therefore $s_0(x) = s_0([(k-1)/b] + (r+s)/b) = s_0((r+s)/b) = u_b(1 + [r+s]) = u_b(1+r) = u_b(k - b[(k-1)/b])$.

COROLLARY 1. $s_n(x) = u_b(k - b[(k-1)/b])$ *for* $(k-1)/b^{n+1} \leqq x < k/b^{n+1}$.

PROOF. $s_n(x) = s_0(b^n x) = u_b(k - b[(k-1)/b])$ for $(k-1)/b \leqq b^n x < k/b$.

COROLLARY 2. $s_n(x)$ *is constant over* $[(k-1)/b^m, k/b^m)$ *when* $m > n$. *In fact*, $s_n(x) = u_b(1 + [(k-1)/b^{m-n-1}] - b[[(k-1)/b^{m-n-1}]/b])$ *on this interval.*

PROOF. Let $k - 1 = b^{m-n-1}[(k-1)/b^{m-n-1}] + r$ where $0 \leqq r < b^{m-n-1}$. Then $k/b^{m-n-1} = [(k-1)/b^{m-n-1}] + (r+1)/b^{m-n-1} \leqq [(k-1)/b^{m-n-1}] + 1$. Therefore

$$[(k-1)/b^{m-n-1}]/b^{n+1} \leqq (k-1)/b^m < k/b^m \leqq ([(k-1)/b^{m-n-1}] + 1)/b^{n+1}$$

and the conclusions follow from Corollary 1.

LEMMA 2. $s_n(x) = s_n(x + 1/b^n)$.

PROOF. $s_n(x + 1/b^n) = s_0(b^n x + 1) = s_0(b^n x) = s_n(x)$.

THEOREM 3. *Let* $f(x)$ *be a power product of the* $s_n(x)$ *in which at least one* $s_n(x)$ *has exponent unity. Then* $\int_0^1 f(x)dx = 0$.

PROOF. Let $f(x) = P(x)Q(x)s_m(x)$ where

$$P(x) = s_{a_1}^{\alpha_1}(x) \cdots s_{a_p}^{\alpha_p}(x),$$

$$Q(x) = s_{c_1}^{\gamma_1}(x) \cdots s_{c_q}^{\gamma_q}(x)$$

and $a_i < m < c_j$ for $1 \leqq i \leqq p$, $1 \leqq j \leqq q$. By Corollary 2 of Lemma 1, $P(x)$ is constant over $[(k-1)/b^m, k/b^m)$. Denote this constant by P_k. By Lemma 2,

$$\int_{(j-1)/b^{m+1}}^{j/b^{m+1}} Q(x)dx = \int_0^{1/b^{m+1}} Q(x)dx \text{ for all } j.$$

Now

$$\int_0^1 P(x)Q(x)s_m(x)dx = \sum_{k=1}^{b^m} \int_{(k-1)/b^m}^{k/b^m} P(x)Q(x)s_m(x)dx$$

$$= \sum_{k=1}^{b^m} P_k \int_{(k-1)/b^m}^{k/b^m} Q(x)s_m(x)dx = \sum_{k=1}^{b^m} P_k \sum_{j=b(k-1)+1}^{bk} \int_{(j-1)/b^{m+1}}^{j/b^{m+1}} Q(x)s_m(x)dx$$

$$= \sum_{k=1}^{b^m} P_k \sum_{j=b(k-1)+1}^{bk} u_b(j - b[(j-1)/b]) \int_{(j-1)/b^{m+1}}^{j/b^{m+1}} Q(x)dx$$

$$= \left(\int_0^{1/b^{m+1}} Q(x)dx\right) \sum_{k=1}^{b^m} P_k \sum_{j=b(k-1)+1}^{bk} u_b(j - b[(j-1)/b])$$

$$= \left(\int_0^{1/b^{m+1}} Q(x)dx\right) \sum_{k=1}^{b^m} P_k \sum_{n=1}^{b} u_b(n).$$

But $\sum_{n=1}^{b} u_b(n) = 0$ by (3) with $P(x) \equiv 1$. This completes the proof.

COROLLARY. *The sequence* $\{t_n(x)\}$ *is orthogonal.*

Unfortunately it is not always true that $\int_0^1 P(x)Q(x)dx = 0$ where $P(x) \neq Q(x)$ are as in the proof of Theorem 3. In fact, if $P(x) = s_0(x)$ and $Q(x) = s_0^2(x)$ then, for $b = 3$,

$$\int_0^1 P(x)Q(x)dx = \int_0^1 s_0^3(x)dx = \int_0^1 (-1)^{3[3x]} \binom{2}{[3x]}^3 dx$$

$$= 1/3 - 8/3 + 1/3 = -2.$$

Hence one cannot complete the functions $s_n(x)$ by the method used in obtaining the Walsh functions of order b from the functions $r_n(x)$. Also since for a given b all functions of the form $P(x)Q(x)s_m(x)$, as in the proof of Theorem 3, are symmetric or antisymmetric about 1/2, according to whether b is odd or even, this set is far from complete. It would be interesting to find the completion of this set of functions.

4. **Normalization of the $t_n(x)$.** In this section we normalize the $t_n(x)$. Our first step is to prove

LEMMA 3.

$$\sum_{i=1}^{bk} u_b^2(i) = \left(\sum_{i=1}^{b} u_b^2(i) \right)^k.$$

PROOF.

$$\sum_{i=1}^{bk} u_b^2(i) = \sum_{j=1}^{bk-1} \sum_{i=(j-1)b+1}^{jb} u_b^2(i) = \sum_{j=1}^{bk-1} \sum_{q=1}^{b} u_b^2((j-1)b+q)$$
$$\overset{*}{=} \sum_{j=1}^{bk-1} \sum_{q=1}^{b} u_b^2(q) u_b^2(j) = \left(\sum_{q=1}^{b} u_b^2(q) \right) \left(\sum_{j=1}^{bk-1} u_b^2(j) \right)$$

and the conclusion follows by induction. At the asterisk we used (5).

LEMMA 4.

$$\int_0^1 t_n^2(x)dx = \left(\binom{2(b-1)}{b-1} \Big/ b \right)^{n+1}.$$

PROOF.

$$\int_0^1 t_n^2(x)dx = \int_0^1 u_b^2 \left(1 + [b^{n+1}x]\right)dx = \sum_{j=1}^{b^{n+1}} \int_{(j-1)/b^{n+1}}^{j/b^{n+1}} u_b^2(j)dx$$
$$= \left(\sum_{i=1}^{b} u_b^2(i) \right)^{n+1} \Big/ b^{n+1} = \left(\frac{1}{b} \sum_{i=1}^{b} u_b^2(i) \right)^{n+1}$$
$$= \left(\frac{1}{b} \sum_{j=0}^{b-1} \binom{b-1}{j}^2 \right)^{n+1} \overset{*}{=} \left(\binom{2(b-1)}{b-1} \Big/ b \right)^{n+1}.$$

At the asterisk we used the binomial identity

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$$

obtained by equating coefficients of x^n in the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$.

Lemma 4 and the results of §3 yield the

COROLLARY. *The sequence*

$$\left\{ \left(b \Big/ \binom{2(b-1)}{b-1} \right)^{(n+1)/2} t_n(x) \right\},$$

$n \geqq 0$, *is an orthonormal sequence.*

5. **A final lemma.** In [2, page 45] it is proved that if $\phi(x)$ is a func-

tion of L^2, of period 1, satisfying $\phi(x)+\phi(x+1/2)=0$ then the sequence $\{\phi_n(x)\}$, $n\geqq 0$, where $\phi_n(x)=\phi(2^n x)$, is orthogonal. Essentially the same proof gives

LEMMA 5. *If $\phi(x)$ is a function of L^2, of period 1, satisfying $\sum_{j=1}^{b}\phi(x+(j-1)/b)=0$ and if $\phi_n(x)=\phi(b^n x)$, $n\geqq 0$, then the sequence $\{\phi_n(x)\}$ is orthogonal on $[0, 1]$.*

PROOF. Let $m>n$. Then

$$\begin{aligned}\int_0^1 \phi_m(t)\phi_n(t)dt &= (1/b^n)\int_0^{b^n}\phi(b^{m-n}u)\phi(u)du = \int_0^1 \phi(b^{m-n}u)\phi(u)du \\ &= \sum_{j=1}^{b}\int_{(j-1)/b}^{j/b}\phi(b^{m-n}u)\phi(u)du \\ &= \sum_{j=1}^{b}\int_0^{1/b}\phi(b^{m-n}(v+(j-1)/b)\phi(v+(j-1)/b)dv \\ &= \sum_{j=1}^{b}\int_0^{1/b}\phi(b^{m-n}v+(j-1)b^{m-n-1})\phi(v+(j-1)/b)dv \\ &= \int_0^{1/b}\phi(b^{m-n}v)\left(\sum_{j=1}^{b}\phi(v+(j-1)/b)\right)dv = 0.\end{aligned}$$

One can prove directly by use of this lemma that the functions $s_n(x)$ are orthogonal. One merely takes $\phi(t)=s_0(t)$. One needs only show that $\sum_{j=1}^{b} s_0(t+(j-1)/b)=0$ and this is not difficult.

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