

$$\frac{k\lambda - (k - 1)}{(k - 1)\lambda - (k - 2)} = \frac{k'\lambda - (k' - 1)}{(k' - 1)\lambda - (k' - 2)}$$

the net result of which is $(k - k')(\lambda - 1)^2 = 0$. Since $0 \leq k < k' \leq n$, and the characteristic p of F is larger than n , $k - k' \not\equiv 0(p)$ and so $(\lambda - 1)^2 = 0$ results. Hence $\lambda = 1$ and the theorem is proved.

We point out that the argument used above also works, up to a point in a Banach algebra. The conclusion one can reach is that the spectrum of $ab^{-1}a^{-1}b$ is invariant under the transformation $(2\lambda - 1)/\lambda$.

BIBLIOGRAPHY

1. C. Putnam and A. Wintner, *On the spectra of commutators*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 360-362.

CORNELL UNIVERSITY

SOME ORTHOGONAL FUNCTIONS CONNECTED WITH POLYNOMIAL IDENTITIES

J. B. ROBERTS

If $P(x)$ is an arbitrary polynomial of sufficiently small degree then there are various ways of choosing an integer N and coefficients c_j such that

$$(1) \quad \sum_{j=0}^N c_j P(x + j) = 0$$

for all x . In each of [3; 4] such an identity is proved. The present paper is devoted to a discussion of a connection between the coefficients in these identities and certain classes of orthonormal functions.

1. **A polynomial identity and the Walsh functions.** In [4] we proved the identity

$$(2) \quad \sum_{n=0}^{b_1^k + 1 - 1} \omega^{v_b(n)} P(x + n) = 0,$$

where b and k are positive integers, $v_b(n)$ is the sum of the coefficients

Presented to the Society, August 30, 1957 under the title *Some classes of orthogonal functions*; received by the editors December 16, 1957.

of n in base b , ω is an arbitrary b th root of unity other than 1, and $P(x)$ is any polynomial of degree $\leq k$. For uniformity in notation with [1] we take $\omega = e^{2\pi i/b}$.

We now define, for each k , a function of x of period 1 whose values are precisely these coefficients on the successive intervals of $[0, 1)$ which are of length $1/b^{k+1}$.

DEFINITION 1.

$$\begin{aligned} w_n(x) &= \omega^{vb([b^{n+1}x])}, & 0 \leq x < 1; \\ w_n(x+1) &= w_n(x) \text{ for all } x & n \geq 0. \end{aligned}$$

Note that these functions depend on b . We shall not make this explicit in our notation as b is to remain fixed.

Our first theorem states that $w_n(x)$ is the product of the first $n+1$ Rademacher functions of order b (see [1]). The Rademacher functions of order b are defined as follows.

$$\begin{aligned} r_0(x) &= \omega^{[bx]}, & 0 \leq x \leq 1; \\ r_n(x+1) &= r_n(x) = r_0(b^n x), & n \geq 0. \end{aligned}$$

THEOREM 1.

$$w_n(x) = \prod_{j=0}^n r_j(x).$$

PROOF. Suppose first that $0 \leq x \leq 1$. Then the coefficient of b^j in the base b expansion of $[b^{n+1}x]$ is congruent modulo b to

$$[[b^{n+1}x]/b^j] = [b^{n-i+1}x].$$

Hence

$$v_b([b^{n+1}x]) \equiv [b^{n+1}x] + [b^n x] + \cdots + [bx] \pmod{b}.$$

Therefore

$$\begin{aligned} \prod_{j=0}^n r_j(x) &= r_0(x)r_0(bx) \cdots r_0(b^n x) \\ &= r_0(x)r_0(bx - [bx]) \cdots r_0(b^n x - [b^n x]) \\ &= \omega^{[bx]}\omega^{[b^2x-b[bx]]} \cdots \omega^{[b^{n+1}x-b[b^n x]]} \\ &= \omega^{[bx]}\omega^{[b^2x]} \cdots \omega^{[b^{n+1}x]} = \omega^{vb([b^{n+1}x])} = w_n(x). \end{aligned}$$

The theorem follows from periodicity.

The Walsh functions of order b (see [1]) are defined by

$$\begin{aligned} \psi_0(x) &= 1; \\ \psi_n(x) &= r_{n_1}^{a_1}(x) \cdots r_{n_k}^{a_k}(x) \end{aligned}$$

where the base b expansion of n is $a_1b^{n_1} + \dots + a_kb^{n_k}$. Hence from Theorem 1 we conclude that the sequence $\{w_n(x)\}$ is an orthonormal set since each $w_n(x)$ is a Walsh function. We also note that $\{w_n(x)\}$ is a lacunary subsequence of $\{\psi_n(x)\}$.

2. **A similar situation with another polynomial identity.** In [3] we proved the identity

$$(3) \quad \sum_{n=0}^{b^k-1} u_b(n+1)P(x+n) = 0$$

where $P(x)$ is a polynomial of degree less than $k(b-1)$ and the coefficients $u_b(n+1)$ are given in (4) below.

$b_j(n)$ is the coefficient of b^j in the base b expansion of n ;

$$(4) \quad \begin{aligned} v_b(n) &= \sum_{j=0}^{\infty} b_j(n); \\ u_b(n+1) &= (-1)^{v_b(n)} \prod_{j=0}^{\infty} \binom{b-1}{b_j(n)}. \end{aligned}$$

An easy consequence of (4) (see [3]) is the identity

$$(5) \quad u_b(n+mb^k) = u_b(n)u_b(m+1) \text{ for } 1 \leq n \leq b^k, \quad m \geq 0.$$

Using the coefficients of (3) we construct a sequence of functions in analogy with the $w_n(x)$.

DEFINITION 2.

$$\begin{aligned} t_n(x) &= u_b(1 + [b^{n+1}x]), \quad 0 \leq x < 1; \\ t_n(x+1) &= t_n(x) \text{ for all } x \quad n \geq 0. \end{aligned}$$

Note that when $b=2$ we have $t_n(x) = w_n(x)$. This case is discussed in [3].

The natural analogues of the $r_n(x)$ out of which one can build the $t_n(x)$ would seem to be the functions $s_n(x)$ defined next.

DEFINITION 3.

$$\begin{aligned} s_0(x) &= u_b(1 + [bx]) = (-1)^{[bx]} \binom{b-1}{[bx]}, \quad 0 \leq x < 1; \\ s_n(x+1) &= s_n(x) = s_0(b^n x), \quad n \geq 0. \end{aligned}$$

THEOREM 2.

$$t_n(x) = \prod_{j=0}^n s_j(x).$$

PROOF. For $n=0$ this is evident. Hence we suppose it is true for

$n \leq k$. Then for $0 \leq x < 1$, $r = b^{k+1}x - [b^{k+1}x]$ we have

$$\begin{aligned} \prod_{j=0}^{k+1} s_j(x) &= t_k(x) s_{k+1}(x) = u_b(1 + [b^{k+1}x]) s_0(b^{k+1}x) \\ &= u_b(1 + [b^{k+1}x]) s_0(r) = u_b(1 + [b^{k+1}x]) u_b(1 + [br]) \\ &\stackrel{*}{=} u_b(1 + [br] + b[b^{k+1}x]) = u_b(1 + [b^{k+2}x]) = t_{n+1}(x). \end{aligned}$$

At the asterisk we used (5). The conclusion now follows from periodicity.

3. Further properties of the $s_n(x)$ and $t_n(x)$ functions. In this section we prove a theorem concerning the orthogonality of power products of the $s_n(x)$. This theorem includes the orthogonality of the $t_n(x)$. We first prove two lemmas.

LEMMA 1. $s_0(x) = u_b(k - [(k-1)/b])$ for $(k-1)/b \leq x < k/b$.

PROOF. Write $k-1 = [(k-1)/b]b + r$, $0 \leq r < b$. Then $x = [(k-1)/b] + (r+s)/b$, $0 \leq s < 1$. Therefore $s_0(x) = s_0([(k-1)/b] + (r+s)/b) = s_0((r+s)/b) = u_b(1 + [r+s]) = u_b(1+r) = u_b(k - b[(k-1)/b])$.

COROLLARY 1. $s_n(x) = u_b(k - b[(k-1)/b])$ for $(k-1)/b^{n+1} \leq x < k/b^{n+1}$.

PROOF. $s_n(x) = s_0(b^n x) = u_b(k - b[(k-1)/b])$ for $(k-1)/b \leq b^n x < k/b$.

COROLLARY 2. $s_n(x)$ is constant over $[(k-1)/b^m, k/b^m]$ when $m > n$. In fact, $s_n(x) = u_b(1 + [(k-1)/b^{m-n-1}]) - b([(k-1)/b^{m-n-1}]/b)$ on this interval.

PROOF. Let $k-1 = b^{m-n-1}[(k-1)/b^{m-n-1}] + r$ where $0 \leq r < b^{m-n-1}$. Then $k/b^{m-n-1} = [(k-1)/b^{m-n-1}] + (r+1)/b^{m-n-1} \leq [(k-1)/b^{m-n-1}] + 1$. Therefore

$$[(k-1)/b^{m-n-1}]/b^{n+1} \leq (k-1)/b^m < k/b^m \leq ([(k-1)/b^{m-n-1}] + 1)/b^{n+1}$$

and the conclusions follow from Corollary 1.

LEMMA 2. $s_n(x) = s_n(x+1/b^n)$.

PROOF. $s_n(x+1/b^n) = s_0(b^n x + 1) = s_0(b^n x) = s_n(x)$.

THEOREM 3. Let $f(x)$ be a power product of the $s_n(x)$ in which at least one $s_n(x)$ has exponent unity. Then $\int_0^1 f(x) dx = 0$.

PROOF. Let $f(x) = P(x)Q(x)s_m(x)$ where

$$P(x) = s_{a_1}^{\alpha_1}(x) \cdots s_{a_p}^{\alpha_p}(x),$$

$$Q(x) = s_{c_1}^{\gamma_1}(x) \cdots s_{c_q}^{\gamma_q}(x)$$

and $a_i < m < c_j$ for $1 \leq i \leq p$, $1 \leq j \leq q$. By Corollary 2 of Lemma 1, $P(x)$ is constant over $[(k-1)/b^m, k/b^m]$. Denote this constant by P_k . By Lemma 2,

$$\int_{(j-1)/b^{m+1}}^{j/b^{m+1}} Q(x)dx = \int_0^{1/b^{m+1}} Q(x)dx \text{ for all } j.$$

Now

$$\begin{aligned} \int_0^1 P(x)Q(x)s_m(x)dx &= \sum_{k=1}^{b^m} \int_{(k-1)/b^m}^{k/b^m} P(x)Q(x)s_m(x)dx \\ &= \sum_{k=1}^{b^m} P_k \int_{(k-1)/b^m}^{k/b^m} Q(x)s_m(x)dx = \sum_{k=1}^{b^m} P_k \sum_{j=b(k-1)+1}^{bk} \int_{(j-1)/b^{m+1}}^{j/b^{m+1}} Q(x)s_m(x)dx \\ &= \sum_{k=1}^{b^m} P_k \sum_{j=b(k-1)+1}^{bk} u_b(j - b[(j-1)/b]) \int_{(j-1)/b^{m+1}}^{j/b^{m+1}} Q(x)dx \\ &= \left(\int_0^{1/b^{m+1}} Q(x)dx \right) \sum_{k=1}^{b^m} P_k \sum_{j=b(k-1)+1}^{bk} u_b(j - b[(j-1)/b]) \\ &= \left(\int_0^{1/b^{m+1}} Q(x)dx \right) \sum_{k=1}^{b^m} P_k \sum_{n=1}^b u_b(n). \end{aligned}$$

But $\sum_{n=1}^b u_b(n) = 0$ by (3) with $P(x) \equiv 1$. This completes the proof.

COROLLARY. *The sequence $\{t_n(x)\}$ is orthogonal.*

Unfortunately it is not always true that $\int_0^1 P(x)Q(x)dx = 0$ where $P(x) \neq Q(x)$ are as in the proof of Theorem 3. In fact, if $P(x) = s_0(x)$ and $Q(x) = s_0^2(x)$ then, for $b = 3$,

$$\begin{aligned} \int_0^1 P(x)Q(x)dx &= \int_0^1 s_0^3(x)dx = \int_0^1 (-1)^{3[3x]} \left(\frac{2}{[3x]} \right)^3 dx \\ &= 1/3 - 8/3 + 1/3 = -2. \end{aligned}$$

Hence one cannot complete the functions $s_n(x)$ by the method used in obtaining the Walsh functions of order b from the functions $r_n(x)$. Also since for a given b all functions of the form $P(x)Q(x)s_m(x)$, as in the proof of Theorem 3, are symmetric or antisymmetric about $1/2$, according to whether b is odd or even, this set is far from complete. It would be interesting to find the completion of this set of functions.

4. Normalization of the $t_n(x)$. In this section we normalize the $t_n(x)$. Our first step is to prove

LEMMA 3.

$$\sum_{i=1}^{bk} u_b^2(i) = \left(\sum_{i=1}^b u_b^2(i) \right)^k.$$

PROOF.

$$\begin{aligned} \sum_{i=1}^{bk} u_b^2(i) &= \sum_{j=1}^{bk-1} \sum_{i=(j-1)b+1}^{jb} u_b^2(i) = \sum_{j=1}^{bk-1} \sum_{q=1}^b u_b^2((j-1)b+q) \\ &= \sum_{j=1}^{bk-1} \sum_{q=1}^b u_b^2(q) u_b^2(j) = \left(\sum_{q=1}^b u_b^2(q) \right) \left(\sum_{j=1}^{bk-1} u_b^2(j) \right) \end{aligned}$$

and the conclusion follows by induction. At the asterisk we used (5).

LEMMA 4.

$$\int_0^1 t_n^2(x) dx = \left(\binom{2(b-1)}{b-1} / b \right)^{n+1}.$$

PROOF.

$$\begin{aligned} \int_0^1 t_n^2(x) dx &= \int_0^1 u_b^2(1 + [b^{n+1}x]) dx = \sum_{j=1}^{b^{n+1}} \int_{(j-1)/b^{n+1}}^{j/b^{n+1}} u_b^2(j) dx \\ &= \left(\sum_{i=1}^b u_b^2(i) \right)^{n+1} / b^{n+1} = \left(\frac{1}{b} \sum_{i=1}^b u_b^2(i) \right)^{n+1} \\ &= \left(\frac{1}{b} \sum_{j=0}^{b-1} \binom{b-1}{j} \right)^{n+1} = \left(\binom{2(b-1)}{b-1} / b \right)^{n+1}. \end{aligned}$$

At the asterisk we used the binomial identity

$$\sum_{j=0}^n \binom{n}{j}^2 = \binom{2n}{n}$$

obtained by equating coefficients of x^n in the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$.

Lemma 4 and the results of §3 yield the

COROLLARY. *The sequence*

$$\left\{ \left(\binom{2(b-1)}{b-1} / b \right)^{(n+1)/2} t_n(x) \right\},$$

$n \geq 0$, is an orthonormal sequence.

5. **A final lemma.** In [2, page 45] it is proved that if $\phi(x)$ is a func-

tion of L^2 , of period 1, satisfying $\phi(x) + \phi(x + 1/2) = 0$ then the sequence $\{\phi_n(x)\}$, $n \geq 0$, where $\phi_n(x) = \phi(2^n x)$, is orthogonal. Essentially the same proof gives

LEMMA 5. If $\phi(x)$ is a function of L^2 , of period 1, satisfying $\sum_{j=1}^b \phi(x + (j-1)/b) = 0$ and if $\phi_n(x) = \phi(b^n x)$, $n \geq 0$, then the sequence $\{\phi_n(x)\}$ is orthogonal on $[0, 1]$.

PROOF. Let $m > n$. Then

$$\begin{aligned} \int_0^1 \phi_m(t)\phi_n(t)dt &= (1/b^n) \int_0^{b^n} \phi(b^{m-n}u)\phi(u)du = \int_0^1 \phi(b^{m-n}u)\phi(u)du \\ &= \sum_{j=1}^b \int_{(j-1)/b}^{j/b} \phi(b^{m-n}u)\phi(u)du \\ &= \sum_{j=1}^b \int_0^{1/b} \phi(b^{m-n}(v + (j-1)/b))\phi(v + (j-1)/b)dv \\ &= \sum_{j=1}^b \int_0^{1/b} \phi(b^{m-n}v + (j-1)b^{m-n-1})\phi(v + (j-1)/b)dv \\ &= \int_0^{1/b} \phi(b^{m-n}v) \left(\sum_{j=1}^b \phi(v + (j-1)/b) \right) dv = 0. \end{aligned}$$

One can prove directly by use of this lemma that the functions $s_n(x)$ are orthogonal. One merely takes $\phi(t) = s_0(t)$. One needs only show that $\sum_{j=1}^b s_0(t + (j-1)/b) = 0$ and this is not difficult.

REFERENCES

1. H. E. Chrestenson, *A class of generalized Walsh functions*, Pacific J. Math. vol. 5 (1955) pp. 17-31.
2. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, New York, 1951.
3. J. B. Roberts, *A curious sequence of signs*, Amer. Math. Monthly vol. 64 (1957) pp. 317-322.
4. ———, *A new proof of a theorem of Lehmer*, Canad. J. Math. (1958) pp. 191-194.

WESLEYAN UNIVERSITY AND
REED COLLEGE