COMMUTATORS IN DIVISION RINGS

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1. **Introduction.** A theorem of Wedderburn [4] states: if every element of a finite dimensional associative algebra is a sum of nilpotent elements, then the algebra is nil. In [1], Kaplansky asked whether this theorem could be generalized to rings with minimum condition, and mentioned that an equivalent question is:

Does there exist a division ring D with $D=[D, D]$? (Here $[D, D]$ denotes the subgroup of the additive group of D generated by all additive commutators $[x, y]=xy-yx$.) An affirmative answer to the second question is equivalent to a negative answer to the first. In the finite dimensional case, the trace function shows that the second question has a negative answer.

In this paper we give an example of a division ring D with $D=[D, D]$. In fact every element of D is itself a commutator, and D_n (the $n\times n$ matrix ring with coefficients in D) has the same property.

2. **Construction of the division ring.** Ore has shown that a noncommutative integral domain in which every two nonzero elements have a nonzero common right multiple can be imbedded in a division ring of (right) fractions [2]. He has also shown that the integral domain of differential polynomials in one variable with coefficients in a division ring has the common right multiple property. We show that a certain ring of differential polynomials in an infinite number of variables also has the common right multiple property, and that the division ring of fraction satisfies $D=[D, D]$ and even the stronger properties mentioned above.

Let F be a field, $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ two infinite sets of variables with the same ordered index set I, and $P=F[\{x_i\}, \{y_i\}]$ the set of formal polynomial expressions

$$\sum a_{n(i_1)\cdots n(i_k)m(j_1)\cdots m(j_l)} x_{i_1}^{n(i_1)} \cdots x_{i_k}^{n(i_k)} y_{j_1}^{m(j_1)} \cdots y_{j_l}^{m(j_l)}$$

where $i_1< \cdots <i_k$, $j_1< \cdots <j_l$ are in I, $a \cdots$ are in F, and only a finite number of terms occur in the sum. Define addition of polynomials the usual way, and multiplication by $[x_i, x_j]=0=[y_i,y_j]$, $[x_i, a]=0=[y_i, a]$ for $a\in F$, $[x_i, y_i]=1$, $[x_i, y_j]=0$ for $i\neq j$.

We show P has the common right multiple property by a method due to Tamari [3]:

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For $p \in P$, let deg (p) denote the total degree of p in the x_i and y_i; then deg $(pq) = \deg p + \deg q$. Let p, q be nonzero polynomials, each of degree $<l$. The problem of finding a common nonzero right multiple $t = pr = qs$ of degree l and such that r and s each contain only those variables that occur in p or in q is the same as the problem of solving a finite number of linear homogeneous equations in a finite number of unknowns: the coefficients of r and s are the unknowns, and each term of $t = pr = qs$ gives an equation. If m is the larger of deg p, deg q and v is the number of variables that occur in p or in q then the number of equations is the same as the number of distinct monomials of degree $\leqq l$ in v variables, i.e. $C_{v+l,v}$, and the number of terms in each of r, s is $\geqq C_{v+l-m,v}$. Thus we have $C_{v+l,v}$ equations in at least $2C_{v+l-m,v}$ unknowns. If $l \geqq m/(1-2^{-1/v})$, there are more unknowns than equations and a nonzero solution exists.

Let D be the division ring of fractions $p/q = pq^{-1}$, p, $q \in P$. If $d = pq^{-1} \in D$, each of p, q contains only a finite number of the variables y_i, and so, for some index n, y_n does not occur in p or in q. Then $[x_n, p] = 0 = [x_n, q]$ and $[x_n, d] = 0$, $[x_n, y_n d] = [x_n, y_n]d = d$. Similarly, if $d_1, \cdots, d_r$ are a finite number of elements of D, then y_n does not occur in any of the d_j for some n, and so $[x_n, y_n d_j] = d_j$, $j = 1, \cdots, r$. In particular if (d_{ij}) is an $m \times m$ matrix over D, we can find n such that $[x_n, y_n d_{ij}] = d_{ij}$, for all i, j. Let $c_{ij} = y_n d_{ij}$ and let (x_n) be the matrix $x_n I$, I the identity matrix; then $[(x_n), (c_{ij})] = (d_{ij})$.

3. **Commutators and nilpotent matrices.** We owe the first proposition and its proof to Professor Kaplansky, and the rest of the section is an amplification of his remarks in [**1**].

PROPOSITION 1. *Let R be any ring, R_n the $n \times n$ matrix ring over R, $(n \geqq 2)$. If $A \in R_n$ and trace of $A \in [R, R]$, then A is a sum of nilpotent elements. In particular, if $A \in [R_n, R_n]$ then A is a sum of nilpotent elements.*

PROOF. Any matrix is the sum of a diagonal matrix (i.e. one with zeros off the main diagonal) and two nilpotent matrices, so that for our purposes only the diagonal elements matter. The following 2×2 matrices are nilpotent:

$$\begin{pmatrix} ab & a \\ -bab & -ba \end{pmatrix} \text{ and } \begin{pmatrix} -ba & -ba \\ ba & ba \end{pmatrix}, \text{ also } \begin{pmatrix} -d & -d \\ d & d \end{pmatrix};$$

thus

$$\begin{pmatrix} [ab] & * \\ * & 0 \end{pmatrix}$$

is a sum of nilpotent matrices.

Now let $A = (a_{ij}) \in R_n$, and $\sum_1^n a_{ii} = c \in [R, R]$, so $a_{11} = c - (a_{22} + \cdots + a_{nn})$. Let xe_{ij} denote the matrix with x in the i, j position and zeros elsewhere. Using the above 2×2 matrices, we see that ce_{11} and $-a_{ii}e_{11} + a_{ii}e_{ii}$, $i \geqq 2$, are sums of nilpotent matrices; therefore this also holds for $a_{11}e_{11} + \cdots + a_{nn}e_{nn}$, and finally for A. If $A \in [R_n, R_n]$, then trace of $A \in [R, R]$.

PROPOSITION 2. *Let D be any division ring, A an element of D_n, $n \geqq 2$. The following conditions are equivalent*:

(a) $A \in [D_n, D_n]$,

(b) *trace of* $A \in [D, D]$,

(c) *A is a sum of nilpotent elements.*

PROOF. We have already shown $a \rightarrow b$, $b \rightarrow c$.

Define a "trace modulo commutators," tr (A), as the coset of the ordinary trace of A in the factor group of the additive group of D modulo $[D, D]$. Then $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$, and $\mathrm{tr}(AB) = \mathrm{tr}\,(BA)$. Now let A be nilpotent: then there exists a nonsingular B such that BAB^{-1} is in Jordan canonical form, i.e. BAB^{-1} is a sum of matrices of the form $n_{k,l} = e_{k+1,k} + e_{k+2,k+1} + \cdots + e_{l+1,l}$, $1 \leqq k \leqq l \leqq n-1$. Finally, $[\sum_{i=k}^{l+1} (i+1-k)e_{i,i}, n_{k,l}] = n_{k,l}$; thus $n_{k,l}$ and also A are in $[R_n, R_n]$. This shows (c)$\rightarrow$(a).

COROLLARY. *If D is a division ring such that $D = [D, D]$, then every element of D_n, $n \geqq 2$, is a sum of nilpotent elements.*

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