RINGS WITH A PIVOTAL MONOMIAL

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1. Introduction. Let $\lambda_1 \cdots \lambda_t$ be a set of indeterminates and let $\pi(\lambda) = \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_d}$ be a monomial of degree d in the λ_i. Let P_π denote the set of all monomials $\sigma(\lambda) = \lambda_{j_1} \cdots \lambda_{j_q}$ such that either $q > d$, or $q \leqq d$ with $j_h \neq i_h$ for some $h \leqq q$. S_π will denote the set of all monomials $\sigma(\lambda)$ of P_π of degree $d(\sigma) = q \geqq d = d(\pi)$.

Generalizing the notion of a polynomial identity of a ring, Drazin has introduced in [2] the idea of a ring with a pivotal monomial. He calls a monomial $\pi(\lambda)$ a *right pivotal monomial* of a ring R if for every substitution $\lambda_i = x_i \in R$, $i = 1, 2, \cdots$, the element $\pi(x) = x_{i_1} \cdots x_{i_d}$ belongs to the *right* ideal generated by the elements $\sigma(x) = x_{j_1} \cdots x_{j_q}$, where $\sigma(\lambda)$ ranges over all monomials of P_π. If this condition holds even if $\sigma(\lambda)$ is restricted to the set S_π, then $\pi(\lambda)$ is said to be a *right strongly pivotal monomial.*

The main structure theorem of rings with pivotal monomials obtained in [2] is that the right primitive rings with a right pivotal monomial are the complete matrix rings D_q of all $q \times q$ matrices over a division ring[1] D, which extends a similar result for rings with identities. The other extremity of types of rings leads naturally to the question whether the structure theory of nil rings with an identity of [1] holds also for rings with a pivotal monomial. This question is answered affirmatively for rings with a strongly pivotal monomial in the first part of the present paper. This result leads to a number of applications; noted among them are the structure theory of algebraic algebras of bounded index and a new proof of a result of Levitzki [4, p. 201] that the nil subring of rings which satisfy the minimum condition for right ideals are nilpotent.

In the second part of the present paper we try to generalize the notion of a pivotal monomial to the utmost, while still keeping valid the structure theory of primitive rings of [2]. This is carried out successfully by introducing the notions of right-quasi-regularity modulo

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[1] The distinction between right primitive and left primitive rings has not been done in [2, Theorem 4] which contains the above quoted result. But in the proof of that theorem the author of [2] says that "we may regard R (the primitive ring) and D as operating on V *from the right,*" which is true only if R is a right primitive ring. Thus the result obtained in [2, Theorem 4] holds for right (or left) primitive rings with a right (left) pivotal monomial. It is still an open problem, if the same is true for *right* primitive rings with a *left* pivotal monomial.

right ideals and Jacobson radicals modulo right ideals. These extensions enable us to characterize completely the rings with the property: that all their primitive representations are complete matrix rings of bounded degree over division rings. Furthermore we show that the right ideals of such rings have the same property.

We first extend slightly the idea of a ring with a strongly pivotal monomial, and we shall use this term throughout this paper in the following sense:

A ring R is a strongly (1) *right PM-ring*; (2) *left PM-ring*; (3) *PM-ring of degree d* if there exists a monomial $\pi(\lambda)=\lambda_{i_1}\cdots\lambda_{i_d}$ such that for every substitution $\lambda_i=x_i\in R$, $i=1, 2, \cdots$ the following holds respectively:

$$(1)\qquad \pi(x)R\subseteq\sum_{\sigma\in S_\pi}\sigma(x)R;\qquad\qquad (2)\qquad R\pi(x)\subseteq\sum_{\sigma\in S_\pi}R\sigma(x);$$

$$(3)\qquad R\pi(x)R\subseteq\sum_{\sigma\in S_\pi}R\sigma(x)R.$$

In view of the proof of [**2**, Theorem 2] we may assume that $\pi(\lambda)$ is linear in each λ_i, hence without loss of generality we may assume that $\pi(\lambda)=\lambda_1\lambda_2\cdots\lambda_d$.

The author is indebted to the referee for many simplifications of notation which clarify immensely the present paper.

2. **Nil subrings of strongly PM-rings.** In what follows $N(R)$ will denote the union of all nilpotent ideals of a ring R.

Following the proof of [**1**, Theorem 1] we show:

THEOREM 1. *If T is a nil subring of a strongly PM-ring R of degree d, then T is locally nilpotent and $T^d\subseteq N(R)$.*

The proof is similar to the proof of [**1**, Theorem 1] (see also [**4**, p. 232]) with some minor changes, which are however important in the proof of the present theorem. Because of these changes we produce here the complete proof.

Let n be an integer $\geqq d+1$. Let $A_i=T^{n-i}RT^i$ for $i=1, 2, \cdots, d$ and $A_i=RT^nR$ for $i>d$. We clearly have:

$$(1)\qquad A_iA_j\subseteq RT^nR\qquad\text{for } i\geqq j.$$

Choose $x_i\in A_i$ for $i\geqq 1$. It follows now that $x_{i_1}\cdot x_{i_2}\cdots x_{i_p}\in RT^nR$ if $p>d$, or if $p=d$ and $(i_1, i_2, \cdots, i_d)\neq(1, 2, \cdots, d)$. Indeed, in the first case either some $i_j\geqq i_{j+1}$ and our results follow by (1), or $i_p>d$ and our result follows by the definition of A_{i_p}; in the second case, if $(i_1, \cdots, i_d)\neq(1, 2, \cdots, d)$ then either $i_j\geqq i_{j+1}$ for some j, or $i_d>d$ which by the same reasoning yields our result.

Now if R is a strongly PM-ring of degree d, then by the remarks preceding our theorem it follows that:

$$Rx_1x_2 \cdots x_dR \subseteq \sum_{\sigma \in S_\pi} R\sigma(x)R.$$

We have just shown that $R\sigma(x)R \subseteq RT^nR$ for $\sigma \in S_\pi$. On the other hand, if x_i ranges over all elements of A_i, the elements $x_1 \cdots x_d$ range over all the elements of $(T^{n-1}RT)(T^{n-2}RT^2) \cdots (T^{n-d}RT^d) = (T^{n-1}R)^dT^d$. Hence

$$(RT^{n-1}R)^{d+1} \subseteq R(T^{n-1}R)^dT^dR \subseteq RT^nR. \tag{2}$$

Consider first the case that T is a nilpotent subring of R, and let k be the minimum integer for which the ideal RT^kR is nilpotent; clearly such an integer k exists. If $k > d$, then (2) implies $RT^{k-1}R$ is also nilpotent, which contradicts the minimality of k. Hence $k \leqq d$, from which one readily deduces that $T^d \subseteq N(R)$.

Let T be a nilring and let $t \in T$. Since the ring $\{t\}$ generated by the element t is nilpotent, it follows by the preceding case that $t^d \in N(R) = N$. Consequently, the quotient ring $(T, N)/N$ is a nil ring of bounded index; hence it follows by [**5**] that $(T, N)/N$ and, therefore, also T are locally nilpotent. Let $t_1, \cdots, t_d$ be d elements of T. Since T is locally nilpotent, the ring $\{t_1, \cdots, t_d\}$ is nilpotent. It follows, therefore, by the preceding case that $\{t_1, t_2, \cdots, t_d\}^d \in N$ and in particular $t_1t_2 \cdots t_d \in N$. This being true for all $t_i \in T$, implies that $T^d \subseteq N$; and, in fact, we have also shown that T is locally nilpotent.

The preceding result immediately implies:

THEOREM 2. *If R is a strongly PM-ring of degree d then*:

(1) *the nil subrings of R/N are nilpotent rings of index $\leqq d$. In particular,*

(2) *If Q is the nil radical of R then $Q^d \subseteq N$. Hence, Q is also the lower radical of R and $Q = N_2(R)$ (for definition see* [**4**, Chapter VIII]).

(3) *If T is a nilpotent subset*[2] *of R, then T^d generates a nilpotent ideal in R.*

It was shown in [**2**] that rings which satisfy the minimum condition for right ideals are strongly PM-rings; and since the radical $N(R)$ of such a ring R is nilpotent, we obtain the following result of Levitzki (see e.g. [**4**, p. 201]):

COROLLARY 1. *The nil subrings of a ring R which satisfy the minimum condition for right ideals are nilpotent.*

[2] The extension of this result to subsets follows readily from the first part of the proof of the preceding theorem. This was pointed out to the author by the referee.

Further applications will be given in the last section.

REMARK. Note the difference between Theorem 1 and the corresponding result for PI-rings of degree d [**1**, Theorem 1]. For PM-rings of degree d, the bound of nilpotency modulo $M(R)$ is $[d/2]$; whereas, for PI-rings of degree d, the bound is d. Simple example of matrix rings shows that in both cases the bounds are the best.

3. **J-pivotal monomials.** In the present section we shall say that an (two sided) ideal P of a ring R is a right-primitive ideal in R, if the quotient ring R/P is a right primitive ring. D will denote an arbitrary division algebra and, as before, D_h will denote the ring of all $h\times h$ matrices with coefficients in D.

The main feature of the right PM-rings R, is that they possess the following property [**2**, Theorem 4]:

(M_d) for every right-primitive ideal P in R, the quotient ring

$$R/P\cong D_h, \quad \text{with} \quad h\leqq d.$$

The converse, namely: that the property (M_d) yields the existence of a right pivotal monomial is not necessarily true. In the present section we shall generalize the notion of a pivotal monomial, so that we shall be able to characterize the rings possessing the property (M_d).

First we extend the notion of quasi-regularity: Let U be a *right* ideal in a ring R. An element $r\in R$ will be said to be *right-quasi-regular modulo* U if there exists $s\in R$ such that $r+s-rs\in U$. A right ideal V is said to be *right-quasi-regular modulo* U if it contains only right quasi-regular elements modulo U.

Let $J(U)$ be the set of all elements $r\in R$ such that for all $x\in R$, rx is right-quasi-regular modulo the right ideal U. Thus $J(0)$ is the Jacobson radical of R.

We shall also use the notation $C(S)$ to denote the intersection of all modular maximal right ideals of R containing a subset S of R.

Parallel to the proof of [**4**, Theorem 2, p. 9] we obtain that:

THEOREM 3. *$C(U)=J(U)$ for every right ideal $U\subseteq R$.*

We shall outline the proof here: if $r\notin J(U)$, then rx is not right-quasi-regular mod U for some $x\in R$, and so the right ideal $(U, (1-rx)R)$ does not contain rx.[3] Hence, it is contained in a modular maximal right ideal U_0. Clearly $rx\notin U_0$ and therefore, $r\notin C(U)\subseteq U_0$. This shows that $C(U)\subseteq J(U)$.

To prove the converse, let $r\in J(U)$. Let U_1 be a modular maximal right ideal containing U and let e be the left identity modulo U_1, i.e.,

[3] The proof holds also for rings without a unit.

$ex-x\in U_1$ for all $x\in R$. If $r\notin U_1$, then since U_1 is maximal it follows that $e=rt+j$ for some $t\in R$ and $j\in U_1$; whence, $e(1-rt)=e(j-e)-(j-e)+j\in U_1$. Now let s be the right-quasi-inverse of rt modulo U, then clearly[3] $e(1-rt)(1-s)\in U_1$, but since $s+rt-rts\in U\subseteq U_1$ it follows that $e(s+rt-rts)\in U_1$ and consequently $e=e(1-rt)(1-s)+e(s+rt-rts)\in U_1$ which is impossible. Hence $r\in U_1$. This being true for all U_1, we have $J(U)\subseteq C(U)$. q.e.d.

REMARK. In the preceding definition and theorem we have used right-quasi-regularity, but clearly a similar definition of left-quasi-regularity modulo a left ideal will lead to a similar result for left ideals; and one can prove that for two sided ideals Q of R, $J(Q)=C(Q)$ will be obtained independently of the right or left approach. In fact one can show that $J(Q)/Q=J(0)$, where $J(0)$ is the Jacobson radical of the quotient in R/Q.

We extend now the idea of a pivotal monomial in the following way: We shall call a monimial $\pi(\lambda)$ a right *J-pivotal monomial* of a ring R if for every substitution $\lambda_i=x_i\in R$, $\pi(x)R\subseteq J(\sum_{\sigma\in P_\pi}\sigma(x)R)$, or equivalently, by the preceding theorem, $\pi(x)r$ is right-quasi-regular mod $\sum\sigma(x)R$ for all $r\in R$.[4] A ring R with a right J-pivotal monomial (JPM) of degree d will be called a *right JPM-ring of degree d.*

Our first result is:

THEOREM 4. *A ring R possesses the property* (M_d) *if and only if R is a right JPM-ring of degree d, and then λ^d is a right J-pivotal monomial of R.*

PROOF. Let $\pi(\lambda)=\lambda_{i_1}\cdots\lambda_{i_d}$ be a right JPM of a ring R. Let P be a right-primitive ideal in R. We may assume that R is a ring of operators acting on the right on a D-vector space M such that R/P is an irreducible ring of endomorphisms of M. We follow now the proof of [2, Theorem 4] and we wish to show that $R/P\cong D_h$, $h\leqq d$, or equivalently that $h=(M:D)\leqq d$. Suppose that $(M:D)>d$, so let $v_0, v_1, \cdots, v_d$ be D-independent elements in M. Since R/P is an irreducible ring of endomorphism of M we can find elements $x_i\in R$ such that

$$v_{j-1}x_i=\overset{i}{\delta}_{ji}x_j \qquad j=1,\cdots,d-1 \text{ and } v_dx_i=0.$$

Then we have $v_0\pi(x)R=v_0x_{i_1}\cdots x_{i_d}R=v_dR\neq 0$ but $v_0\sigma(x)R=0$ for all $\sigma\in P_\pi$. Let $U_0=\{r;v_0r=0\}$; then clearly U_0 is a maximal modular right ideal in R, and we thus have shown $\pi(x)R\not\subseteq U_0$, whereas, $\sigma(x)R\subseteq U_0$. Hence clearly[4] $\pi(x)R\not\subseteq J(\sum\sigma(x)R)\subset U_0$ by the preceding theorem, and this contradicts the definition of a JPM.

[4] Where the sum ranges over all $\sigma\in P_\pi$.

Conversely, let R be a ring satisfying (M_d). We shall prove that λ^d is a right JPM of R. First we prove that the rings D_h have the right (and left) pivotal monomial λ^d. Indeed, for $x \in D_h$ the sequence of right ideals: $xD_h \supseteq x^2D_h \supseteq \cdots \supseteq x^dD_h \supseteq x^{d+1}D_h$ cannot contain more than d different ideals, since the length of the composition series of right ideals of D_h is $h \leqq d$. Hence, we must have $x^{d+1}D_h = x^dD_h$, i.e., $x^d \in x^{d+1}D_h$, which proves our assertion.[5]

We shall now show that $\pi(\lambda) = \lambda_1^d$ is a JPM of R. Indeed, let $\lambda_i = x_i \in R$ and let U be a maximal modular right ideal containing[4] $J(\sum \sigma(x)R)$ and let P be the right-primitive ideal contained in U. Since $R/P \cong D_h$, $h \leqq d$, it follows that $x_1^dR \subseteq (\sum \sigma(x)R, P) \subseteq (\sum \sigma(x)R, U)$. This being true for all $U \supseteq J(\sum \sigma(x)R)$ implies, by Theorem 5, that $x_1^dR \subseteq J(\sum \sigma(x)R)$. q.e.d.

Actually our method yields more:

Corollary 2. *If R satisfies (M_d) then the right JPM of all matrix rings D_h, $h \leqq d$ is also a right JPM of R.*

Corollary 3. *If R satisfies (M_d) then $x^d \in J(x^{d+1}R)$ for every $x \in R$.*

4. Right ideals of JPM-rings. We intend to show in the present section that:

Theorem 5. *If U is a right ideal in a JPM-ring R of degree d, then U is also a JPM-ring of degree $\leqq d$.*

To this end we shall need the following lemma:

Lemma 1. *If R is a JPM-ring of degree d, then for every $x \in R$ the right ideal x^dR is right-quasi-regular mod $x^{d+2}R$.*

The proof of this lemma follows the same methods which were used in the proof of Theorem 4. Indeed, for every matrix ring D_h, $h \leqq d$, we have seen that $x^dD_h = x^{d+1}D_h$, $x \in D_h$, whence we also have $x^dD_h = x^{d+2}D_h$. By repeating now the last argument of the proof of the preceding theorem, one obtains that $x^dR \subseteq J(x^{d+2}R)$ for $x \in R$, and the rest follows now by Theorem 3.

We are now ready to prove Theorem 5.

Let $x \in U$. The preceding lemma implies that x^dR is right-quasi-regular modulo $x^{d+2}R$. From the facts that $x^dU \subseteq x^dR$ and that $x^{d+2}R \subseteq x^{d+1}U$, one readily deduces that x^dU is right-quasi-regular modulo $x^{d+1}U$. Our theorem will follow now by showing that for every x^dr, $r \in U$, its right-quasi-inverse s modulo $x^{d+1}U$ belongs to U. Indeed, since $x^dr + s - x^drs = x^{d+1}t \in x^{d+1}U$, it follows that $s = x^{d+1}t - x^dr + x^drs \in U$. This readily implies that λ^d is a right JPM of U.

[5] Compare with the proof of [2, Theorem 4].

It follows now by Theorem 4 that:

COROLLARY 4. *If a ring R possesses the property* (M_d) *then the right ideals of R possess the same property.*

In particular, if U is a right ideal of a semi-simple JPM-ring R then it is known [**4**, Proposition 2, p. 10] that the right anihilator U_0 of U in U is the Jacobson radical of U. Hence, we obtain the following extension of the property of matrix rings over division rings:

COROLLARY 5. *If U is a right ideal in a semi-simple JPM-ring of degree d then the quotient ring U/U_0 is a semi-simple ring satisfying* (M_d).

5. **Concluding remarks and applications.** In the preceding section we have used the property of quasi-regularity to define a J-pivotal monomial of a ring, but clearly many other properties can be used and one will obtain other types of pivotal monomials, as the following example shows:

Call a monomial $\pi(\lambda)$ a(right) *nilpotent-pivotal-monomial* of a ring R if for every substitution $\lambda_i = x_i \in R$ the ideal $\pi(x)R$ is a nilpotent right ideal modulo the right ideal[4] $\sum \sigma(x)R$. A ring with such a monomial will be called an NPM-ring. It is not difficult to show that an NPM-ring is a JPM-ring (to prove this one has only to assume that the right ideal $\pi(x)R$ contains only nilpotent elements mod $\sum \sigma(x)R$). Furthermore, if one defines a strongly NPM-ring in the obvious way one can show that the results of §2 will remain true also for these rings.

We conclude this paper with an application of the theory of §2 to algebraic algebras of bounded index, and with another result on PM-rings.

First we note that

THEOREM 6. *If R is an algebraic algebra of index $\leqq d$, then R is a strongly right PM-ring of degree $\leqq d$.*

Indeed, for $r \in R$ let $g(x) = x^k(x^m + \alpha_1 x^{m-1} + \cdots + \alpha_m) = x^k f(x)$, with $\alpha_m \neq 0$, be the minimal equation of r. Clearly $rf(r)$ is a nil element in R, hence $r^d f(r)^d = 0$ which clearly implies that $r^d R \in r^{d+1}R$.[6] q.e.d.

Hence we obtain by [**2**, Theorem 4] and by the preceding results that:

COROLLARY 6. (1) *If R is a primitive algebraic algebra of index d then $R = D_d$ where D is an algebraic division algebra.*[7]

[6] The present simple proof is due to Dr. Drazin.

[7] Theorem 2 of [**4**, Chapter X, p. 237].

(2) *If R is an algebraic algebra of index $\leqq d$ and Q is the lower radical of R, then R/Q is a subdirect sum of matrix rings D_q, with $q \leqq d$; furthermore $Q^d \subseteq N(R)$.*

(3) *If R is a semi-simple algebraic algebra (in the sense of Jacobson) of degree q, then the nil subrings of R are nilpotent subrings of index $\leqq q$.*

In the preceding section we have shown that the matrix ring D_q is a strongly PM-ring of degree q, hence (3) of Theorem 2 implies that:

COROLLARY 7. *The nil subrings of D_q are nilpotent, and the nilpotent subsets T of D_q satisfy $T^q = 0$.*

Another result on PM-rings is the following:

THEOREM 7. *If R is a PM-ring with a unit, then the right (left) inverses of the elements of R are also left (right) inverses.*

Indeed, if $uv = 1$ $u, v \in R$, then it was shown in [**3**] that if $vu \neq 1$, then R contains nonzero elements c_{ij}, $i, j \geqq 1$, satisfying $c_{ik}c_{jh} = \delta_{kj}c_{ih}$. Now set $x_1 = c_{12}$, $x_2 = c_{23} \cdots x_d = c_{dd+1}$; then $x_1 x_2 \cdots x_d = c_{1d+1}$ but any other product $x_{j_1} \cdots x_{j_q} = 0$. Hence, if R is a PM-ring we obtain

$$0 \neq c_{1d+1} \in Rx_1 \cdots x_d R \subseteq \sum_{\sigma \in S_\pi} R\sigma(x)R = 0,$$

which is a contradiction.

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