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SIMPLE NODAL NONCOMMUTATIVE JORDAN ALGEBRAS

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1. Introduction. Nodal algebras were defined by R. D. Schafer [**4**] and have also been studied by the author [**2**; **3**]. A noncommutative Jordan algebra is an algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ satisfying (1) the flexible law $(xy)x=x(yx)$ and (2) the condition that $\mathfrak{A}^+$ is a Jordan algebra. That is, $\mathfrak{A}^+$ satisfies the identity $(x^2\cdot y)\cdot x=x^2\cdot(y\cdot x)$ where we have used the dot to indicate the product of $\mathfrak{A}^+$. The algebra $\mathfrak{A}^+$ is defined to be the same vector space as $\mathfrak{A}$ but with product $x\cdot y=(xy+yx)/2$ where xy and yx are products in $\mathfrak{A}$. Then $\mathfrak{A}$ is called nodal if it is finite dimensional, if $\mathfrak{A}$ has identity element 1, if $\mathfrak{A}$ can be written as a vector space direct sum $\mathfrak{A}=\mathfrak{F}1+\mathfrak{N}$ where $\mathfrak{N}$ is the subspace of nilpotent elements of $\mathfrak{A}$, and if $\mathfrak{N}$ is not a subalgebra of $\mathfrak{A}$.

Every known nodal algebra $\mathfrak{A}$ has the property that $\mathfrak{A}^+$ is an associative algebra. The flexible algebras with $\mathfrak{A}^+$ associative have been described in [**3**]. In this paper we shall prove the following theorem.

Theorem 1. *Let $\mathfrak{A}$ be a simple nodal noncommutative Jordan algebra of characteristic $\neq 2$. Then $\mathfrak{A}^+$ is associative.*

Define $\mathfrak{B}$ to be the subspace of $\mathfrak{A}$ generated by all the associators in $\mathfrak{N}^+$. That is, $\mathfrak{B}$ is generated by elements of the form $(x\cdot y)\cdot z-x\cdot(y\cdot z)$ with x, y, z in $\mathfrak{N}$. The proof of the theorem will be made by showing that the ideal $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\mathfrak{B}$ is not all of $\mathfrak{A}$ and since $\mathfrak{A}$ is simple it will follow that $\mathfrak{C}=0$ and $\mathfrak{B}=0$. This is the desired result.

The original proof was not valid when the characteristic is 3. The author thanks Professor R. D. Schafer for suggesting a modification

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which makes the proof simpler and also valid when the characteristic is 3.

2. **The proof.** Let x_1, x_2, x_3, and y be any elements of $\mathfrak{N}$. Since $\mathfrak{A}$ is nodal we have:

$$x_i y = \lambda_i 1 + z_i. \tag{1}$$

The proof will depend on relation (8) of Schafer's paper [**4**] which is

$$\begin{aligned}(x_1\cdot x_2)y = {} & \lambda_1 x_2 + \lambda_2 x_1 + x_1\cdot z_2 + x_2\cdot z_1 \\ & - (x_1\cdot y)\cdot x_2 - (x_2\cdot y)\cdot x_1 + (x_1\cdot x_2)\cdot y.\end{aligned} \tag{2}$$

In the proof it will also be necessary to use the fact that $\mathfrak{N}^+$ is a subalgebra of $\mathfrak{A}^+$ [**1**].

By (2) $(x_1\cdot x_2)y$ is in $\mathfrak{N}$ and it follows from (2) that $[(x_1\cdot x_2)\cdot x_3]y = \lambda_3 x_1\cdot x_2 + (x_1\cdot x_2)\cdot z_3 + x_3\cdot[\lambda_1 x_2 + \lambda_2 x_1 + x_1\cdot z_2 + x_2\cdot z_1 - (x_1\cdot y)\cdot x_2 - (x_2\cdot y)\cdot x_1 + (x_1\cdot x_2)\cdot y] - [(x_1\cdot x_2)\cdot y]\cdot x_3 - (x_3\cdot y)\cdot(x_1\cdot x_2) + [(x_1\cdot x_2)\cdot x_3]\cdot y$. Without bothering to simplify interchange subscripts 1 and 3 to get $[x_1\cdot(x_2\cdot x_3)]y = \lambda_1 x_3\cdot x_2 + (x_3\cdot x_2)\cdot z_1 + x_1\cdot[\lambda_3 x_2 + \lambda_2 x_3 + x_3\cdot z_2 + x_2\cdot z_3 - (x_3\cdot y)\cdot x_2 - (x_2\cdot y)\cdot x_3 - (x_1\cdot y)\cdot(x_3\cdot x_2) + [(x_3\cdot x_2)\cdot x_1]\cdot y$. Using the notation $(a, b, c) = (a\cdot b)\cdot c - a\cdot(b\cdot c)$ for the associator of a, b, c in $\mathfrak{A}^+$ we have, upon subtracting the second relation from the first, $(x_1, x_2, x_3)y = (x_1, x_2, z_3) + (x_1, z_2, x_3) + (z_1, x_2, x_3) - (x_1\cdot y, x_2, x_3) - (x_1, x_2\cdot y, x_3) + (x_3\cdot y, x_2, x_1) + (x_1, x_2, x_3)\cdot y$.

Now define the set $\mathfrak{B}$ to be the subspace of $\mathfrak{A}$ generated by the associators (a, b, c) with a, b, c in $\mathfrak{N}$ and using the product of $\mathfrak{N}^+$. We have proved the following lemma.

LEMMA 1. *Let $\mathfrak{A}$ be a nodal noncommutative Jordan algebra whose characteristic is not* 2. *Then* $\mathfrak{B}\mathfrak{N} \subseteq \mathfrak{B} + \mathfrak{B}\cdot\mathfrak{N}$. *Also* $\mathfrak{N}\mathfrak{B} \subseteq \mathfrak{B} + \mathfrak{B}\cdot\mathfrak{N}$.

The last statement follows from the fact that if b is in $\mathfrak{B}$, n in $\mathfrak{N}$, then $nb = 2b\cdot n - bn$.

Let $\mathfrak{C}_0 = \mathfrak{B}$, $\mathfrak{C}_1 = \mathfrak{B} + \mathfrak{B}\cdot\mathfrak{N} = \mathfrak{C}_0 + \mathfrak{C}_0\cdot\mathfrak{N}$, and in general $\mathfrak{C}_{i+1} = \mathfrak{C}_i + \mathfrak{C}_i\cdot\mathfrak{N}$. Equivalently, $\mathfrak{C}_{i+1} = \mathfrak{C}_i + \mathfrak{B}(R^+_{\mathfrak{N}})^{i+1}$.

LEMMA 2. *The product* $(\mathfrak{B}\cdot\mathfrak{N})\mathfrak{N} \subseteq \mathfrak{C}_2$ *and* $\mathfrak{N}(\mathfrak{B}\cdot\mathfrak{N}) \subseteq \mathfrak{C}_2$. *It follows that* $\mathfrak{C}_1\mathfrak{N} \subseteq \mathfrak{C}_2$, $\mathfrak{N}\mathfrak{C}_1 \subseteq \mathfrak{C}_2$.

The proof follows from the flexible law as does (2) which was proved by Schafer. The linearized form of the flexible identity is

$$(xy)z + (zy)x = x(yz) + z(yx). \tag{3}$$

Add $(yx)z + (yz)x$ to both sides of (3) to obtain the equivalent relation

$$(x\cdot y)z + (y\cdot z)x = yz\cdot x + yx\cdot z. \tag{4}$$

If x is in $\mathfrak{B}$, y, z in $\mathfrak{N}$, then $(y \cdot z)x$ is in $(\mathfrak{N} \cdot \mathfrak{N})\mathfrak{B} \subseteq \mathfrak{N}\mathfrak{B}$. By Lemma 1, $(y \cdot z)x$ is in $\mathfrak{C}_1$. The product yz is in $\mathfrak{F}1 + \mathfrak{N}$ so $yz \cdot x$ is in $\mathfrak{B} + \mathfrak{N} \cdot \mathfrak{B} = \mathfrak{C}_1$. And $yx \cdot z$ is in $\mathfrak{N}\mathfrak{B} \cdot \mathfrak{N} \subseteq \mathfrak{C}_2$. Therefore, $(x \cdot y)z$ is in $\mathfrak{C}_2$ as desired.

LEMMA 3. *The product* $[\mathfrak{B}(R^+_{\mathfrak{N}})^i]\mathfrak{N} \subseteq \mathfrak{C}_{i+1}$ *and* $\mathfrak{N}[\mathfrak{B}(R^+_{\mathfrak{N}})^i] \subseteq \mathfrak{C}_{i+1}$. *Or, equivalently,* $\mathfrak{C}_i\mathfrak{N} \subseteq \mathfrak{C}_{i+1}$, $\mathfrak{N}\mathfrak{C}_i \subseteq \mathfrak{C}_{i+1}$.

Assume that $[\mathfrak{B}(R^+_{\mathfrak{N}})^{i-1}]\mathfrak{N}$ and $\mathfrak{N}[\mathfrak{B}(R^+_{\mathfrak{N}})^{i-1}]$ are in $\mathfrak{C}_i$. Take x in (4) to be in $\mathfrak{H} = \mathfrak{B}(R^+_{\mathfrak{N}})^{i-1}$, and y, z to be in $\mathfrak{N}$. Then $(y \cdot z)x$ is in $\mathfrak{N}\mathfrak{H} \subseteq \mathfrak{C}_i$, $yz \cdot x$ is in $\mathfrak{H} + \mathfrak{N} \cdot \mathfrak{H} \subseteq \mathfrak{C}_i$, and $yx \cdot z$ is in $(\mathfrak{N}\mathfrak{H}) \cdot \mathfrak{N} \subseteq \mathfrak{C}_{i+1}$. Thus $(x \cdot y)z$ is in $\mathfrak{C}_{i+1}$.

LEMMA 4. *There exists a positive integer* k *such that* $\mathfrak{C}_k = \mathfrak{C}_{k+1}$ *and* $\mathfrak{C}_k$ *is an ideal of* $\mathfrak{A}$.

The set $\mathfrak{B}$ is contained in $\mathfrak{N}$. Since $\mathfrak{A}^+$ is a Jordan algebra, $\mathfrak{N}^+$ is nilpotent. Consequently, $\mathfrak{B}(R^+_{\mathfrak{N}})^{k+1} = 0$ for some k. For this k, $\mathfrak{C}_k = \mathfrak{C}_{k+1}$. By Lemma 3 $\mathfrak{C}_k\mathfrak{N} \subseteq \mathfrak{C}_{k+1} = \mathfrak{C}_k$ and $\mathfrak{N}\mathfrak{C}_k \subseteq \mathfrak{C}_k$.

The ideal $\mathfrak{C}_k$ is contained in $\mathfrak{N} \cdot \mathfrak{N} \cdot \mathfrak{N} = \mathfrak{N}_3$. Since $\mathfrak{N}^+$ is a subalgebra of $\mathfrak{A}^+$, $\mathfrak{N}_3 \subset \mathfrak{N}$ and so $\mathfrak{C}_k \subset \mathfrak{N}$. If $\mathfrak{A}$ is a simple algebra, $\mathfrak{C}_k = 0$ and thus $\mathfrak{B} = 0$. This says that every associator in $\mathfrak{N}^+$ is zero. Now if a, b, c are any elements in $\mathfrak{A}$, $a = \alpha 1 + x$, $b = \beta 1 + y$, $c = \gamma 1 + z$ with x, y, z in $\mathfrak{N}$. Then $(a \cdot b) \cdot c - a \cdot (b \cdot c) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ so every associator in $\mathfrak{A}^+$ is an associator in $\mathfrak{N}^+$. This completes the proof of the theorem.

Any ideal properly contained in $\mathfrak{A}$ is contained in $\mathfrak{N}$, hence is a nilideal and is contained in the radical of $\mathfrak{A}$. This implies the corollary which we state below.

COROLLARY. *Let* $\mathfrak{A}$ *be a semisimple nodal noncommutative Jordan algebra of characteristic* $\neq 2$. *Then* $\mathfrak{A}$ *is simple and* $\mathfrak{A}^+$ *is associative.*

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