Let $A = \|a_{i,j}\|$ be an $n \times n$ matrix consisting of non-negative elements. It is well known [1, p. 463] that $A$ is primitive if and only if, for some positive integer $n$, $A^n$ has all its elements positive. One needs to know only this property of primitive matrices to understand this paper. If $A^k$ is positive (i.e. has all its elements positive), then $A^h$ is also positive for all integers $h > k$ [1, p. 463].\(^2\) Letting $A$ be primitive, we shall define $\gamma(A)$ as the smallest positive integer $h$ such that $A^h$ is positive.

Wielandt [2, p. 648] stated without proof the inequality\(^3\)

\[ \gamma(A) \leq n^2 - 2n + 2, \]

and gave an example to show that $\gamma(A)$ could equal $n^2 - 2n + 2$. In the special case that all the diagonal elements of $A$ are positive, Wielandt [2, p. 644] showed that one may obtain the better bound

\[ \gamma(A) \leq n - 1. \]

In this paper, we show that when there are one or more positive diagonal elements of $A$ (or of one of its low order powers), bounds may be found for $\gamma(A)$ which are better than (1), although not necessarily as good as (2). We shall also give an easy proof of (1).

In our discussion, we shall assume that the matrix $A$ is non-negative and primitive.\(^4\) Let $J$ be the set of positive integers one through $n$. For $L$ a subset of $J$, define $F^0(L) = L$ and, by induction, for $h$ a positive integer, define $F^h(L)$ as the set of all $i \in J$ such that for some $j \in F^{h-1}(L)$, $a_{i,j} > 0$. For $h$ a non-negative integer, and $j \in J$, define $F^h(j)$ as $F^h(L)$ where $L$ is the set containing $j$ and only $j$. We remark that, for $h$ a positive integer, the element of $A^h$ in the $i$th row and $j$th column is positive if and only if $i \in F^h(j)$.

**Lemma 1.** $F(J) = J$.
Lemma 2. If \( L \) is a proper subset of \( J \), then \( F(L) \) contains some element not in \( L \).

Proof. If not, then \( J \supseteq L \supseteq F(L) \supseteq \cdots \supseteq F^{r}(A)(L) = J \) which contradicts \( J \not\supseteq L \).

Corollary. If \( h \leq n - 1 \), then \( \{j\} \cup F(j) \cup \cdots \cup F^{h}(j) \) contains at least \( h + 1 \) elements.

Proof. This is obviously true for \( h = 0 \). Using mathematical induction, assume it is true for some \( 0 \leq h \leq n - 1 \). Set \( L = \{j\} \cup \cdots \cup F^{h}(j) \), and apply Lemma 2.

We remark that, given \( j \in J \), the set of integers \( h \) such that \( j \in F^{h}(j) \) is a semigroup. Therefore, properties described below may be easily observed by observing the first few iterates of \( A \).

Lemma 3. Let \( k \) be a non-negative integer, and \( j \in J \). For \( h \geq k \), let \( j \in F^{h}(j) \). Then, \( F^{n-k}(j) = J \).

Proof. The corollary above implies that \( \{j\} \cup \cdots \cup F^{n-1}(j) = J \). For each \( 0 \leq h \leq n - 1 \), \( j \in F^{n-1+k-h}(j) \), and so \( F^{h}(j) \subseteq F^{n-1+k}(j) \). Therefore, \( J = \bigcup_{n=0}^{n-1} F^{h}(j) \subseteq F^{n-1+k}(j) \subseteq J \).

Theorem 1. Let \( k \) be a non-negative integer. Let there be at least \( d > 0 \) elements \( j \) of \( J \) such that for \( h \leq k \), the \( j \)th diagonal element of \( A^{h} \) is positive. Then, \( \gamma(A) \leq 2n - d - 1 + k \).

Proof. The corollary above implies that, for each \( j \in J \), there exists \( 0 \leq h \leq n - d \) such that \( F^{h}(j) \) contains at least one of the \( d \) elements described above. Then,

\[
J \supseteq F^{2n-d-1+k}(j) = F^{n-d-h} \left\{ F^{n-1+k} \left[ F^{h}(j) \right] \right\} \supseteq F^{n-d-h}(J) = J.
\]

Corollary. Let \( d > 0 \) of the diagonal elements of \( A \) be positive. Then, \( \gamma(A) \leq 2n - d - 1 \).

Theorem 2. Let \( h \) be a positive integer, and let \( A + A^{2} + \cdots + A^{h} \) have at least \( d > 0 \) of its diagonal elements positive. Then, \( \gamma(A) \leq n - d + h(n-1) \).

Proof. Let \( j \) be one of the \( d \) elements such that \( j \in F^{p}(j) \) for some \( p, 1 \leq p \leq h \). Then, if we substitute 0 for \( k \), and \( F^{p} \) for \( F \), we may apply Lemma 3, and conclude that \( F^{(n-1)p}(j) = J \). Choose arbitrarily \( j' \subseteq J \). Then, the corollary to Lemma 2 implies that there exists an \( l \),\(^6\) if all the diagonal elements of \( A \) are positive, then \( d = n \), and the inequality of the corollary reduces to Wielandt's result (2).
0 \leq l \leq n - d \) such that \( F^l(j') \) contains at least one of these \( d \) elements. Therefore, \( J \supseteq F^{n-d+l}(n-1)(j') = F^{n-d-l+(h-p)}(n-1) \{ F^p(n-1) [ F^l(j') ] \} \supseteq F^{n-d-l+(h-p)}(n-1)(J) = J \), since \( n-d-l+(h-p)(n-1) \geq 0 \).

**Corollary.** Let \( A \) be non-negative and positively symmetric in that \( a_{i,j} > 0 \) if and only if \( a_{j,i} > 0 \). Then, \( \gamma(A) \leq 2(n-1) \).

**Proof.** \( A^2 \) has all its diagonal elements positive. Now, apply Theorem 2.

**Theorem 3.** \( \gamma(A) \leq n^2 - 2n + 2 \).

**Proof.** Given \( j \in J \), consider the case where \( \{ j \} \cup \cdots \cup F^{n-2}(j) \neq J \). Then, for \( 1 \leq h \leq n-1 \), \( F^h(j) \) contains exactly one element not in \( \{ j \} \cup \cdots \cup F^{h-1}(j) \). Let \( p \) be the smallest positive integer such that \( F^p(j) \) contains at least two elements. Then, there exists an integer \( m < p \) such that \( m > 0 \) (unless \( p = 1 \), in which case \( m = 0 \)) and such that \( F^m(j) \subseteq F^p(j) = F^{m+(p-m)}(j) \subseteq F^{m+2(p-m)}(j) \subseteq \cdots \). Lemma 2 implies that \( \gamma^m(j) = \gamma(j) = \gamma^m+(p-m)(j) = \gamma^m+(n-1)(p-m)(j) = J \). But \( p \leq n \) implies that

\[
m + (n - 1)(p - m) = p + (n - 2)(p - m) \leq n^2 - 2n + 2.
\]

If \( \{ j \} \cup \cdots \cup F^{n-2}(j) = J \), then there exists an integer \( h \), \( 0 \leq h \leq n - 1 \), such that \( F^0(j) \subseteq F^h(j) \subseteq \cdots \subseteq F^{(n-1)h}(j) = J \). But, \( (n-1)h \leq n^2 - 2n + 1 < n^2 - 2n + 2 \). This completes the proof.

Let \( A \) and \( B \) be two non-negative primitive matrices such that if \( A = [a_{i,j}] \), and \( B = [b_{i,j}] \), then \( a_{i,j} > 0 \) implies that \( b_{i,j} > 0 \). It is clear that \( \gamma(A) \geq \gamma(B) \). Furthermore, if \( B \) has many positive elements for which there are no corresponding positive elements of \( A \), then one would expect to have \( \gamma(A) > \gamma(B) \). We shall show that when there are many positive off-diagonal elements of a non-negative primitive matrix, some of the preceding inequalities may be improved.

Given a positive integer \( j \), \( 1 \leq j \leq n \), define \( X(j) \) as the number of elements \( a_{i,j} \), \( i \neq j \), for which \( a_{i,j} > 0 \). Then, the corollary to Lemma 2 implies that \( X(j) \geq 1 \) whenever \( n > 1 \), for all \( j \). Whenever \( X(j) > 1 \), we may improve the result of the corollary to Lemma 2 by observing that if \( 1 \leq h \leq n - X(j) \), then \( \{ j \} \cup F^h(j) \cup \cdots \cup F^h(j) \) contains at least \( h + X(j) \) elements. If we use this result in the proofs of Lemma 3 and Theorem 1, we obtain the following improvements.

**Lemma 4.** Let \( k \) and \( j \) be as in Lemma 3. Then, \( F^{n-X(j)+k}(j) = J \).

**Theorem 4.** Let \( A \) be as in Theorem 1. Let \( X_1 \) be the minimum of \( X(j) \) for the \( d \) elements \( j \in J \). Let \( X_2 \) be the minimum of \( X(j) \) for the remaining \( n-d \) elements \( j \in J \). Then,

\[
\gamma(A) \leq 2n - d - X_1 - \min \{ X_2 - 1; n - d \} + k.
\]
**Corollary.** Let \( d > 0 \) of the diagonal elements of \( A \) be positive. Then,
\[
\gamma(A) \leq 2n - d - X_1 - \min [X_2 - 1; n - d].
\]
A similar improvement may also be obtained for Theorem 2.

For any non-negative irreducible matrix, we may define the (irreducible) order of \( A \), denoted by \( \Lambda(A) \), as the smallest positive integer \( h \) such that \( I + A + A^2 + \cdots + A^h \) is positive, or equivalently, \( \{j\} \cup \cdots \cup F^h(j) = J \) for each \( j \). By definition of irreducibility, it is clear that \( \Lambda(A) \leq n - 1 \). If \( \Lambda(A) \) is less than \( n - 1 \), and the value of \( \Lambda(A) \) is known, many of the preceding inequalities may be improved.

We summarize how the order of \( A \) may be used to sharpen respectively the results of Lemma 4, Theorem 4, and its corollary above. These results are respectively:

1. \[
\gamma(A) \leq \min [n - X_1; \Lambda(A)]
\]
2. \[
\gamma(A) \leq \min [n - X_1; \Lambda(A)] + \min \{n - d - \min [X_2 - 1; n - d]; \Lambda(A)\} + k,
\]
3. \[
\gamma(A) \leq \min [n - X_1; \Lambda(A)] + \min \{n - d - \min [X_2 - 1; n - d]; \Lambda(A)\}.
\]

**Bibliography**


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