# ON POWERS OF NON-NEGATIVE MATRICES

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Let $A = \|a_{i,j}\|$ be an $n \times n$ matrix consisting of non-negative elements. It is well known [**1**, p. 463] that $A$ is *primitive* if and only if, for some positive integer $n$, $A^n$ has all its elements positive. One needs to know only this property of primitive matrices to understand this paper. If $A^k$ is positive (i.e. has all its elements positive), then $A^h$ is also positive for all integers $h > k$ [**1**, p. 463].[2] Letting $A$ be primitive, we shall define $\gamma(A)$ as the smallest positive integer $h$ such that $A^h$ is positive.

Wielandt [**2**, p. 648] stated without proof the inequality[3]

$$\gamma(A) \leqq n^2 - 2n + 2, \tag{1}$$

and gave an example to show that $\gamma(A)$ could equal $n^2 - 2n + 2$. In the special case that all the diagonal elements of $A$ are positive, Wielandt [**2**, p. 644] showed that one may obtain the better bound

$$\gamma(A) \leqq n - 1. \tag{2}$$

In this paper, we show that when there are one or more positive diagonal elements of $A$ (or of one of its low order powers), bounds may be found for $\gamma(A)$ which are better than (1), although not necessarily as good as (2). We shall also give an easy proof of (1).

In our discussion, we shall assume that the matrix $A$ is non-negative and primitive.[4] Let $J$ be the set of positive integers one through $n$. For $L$ a subset of $J$, define $F^0(L) = L$ and, by induction, for $h$ a positive integer, define $F^h(L)$ as the set of all $i \in J$ such that for some $j \in F^{h-1}(L)$, $a_{i,j} > 0$. For $h$ a non-negative integer, and $j \in J$, define $F^h(j)$ as $F^h(L)$ where $L$ is the set containing $j$ and only $j$. We remark that, for $h$ a positive integer, the element of $A^h$ in the $i$th row and $j$th column is positive if and only if $i \in F^h(j)$.

LEMMA 1. $F(J) = J$.

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[2] One may also use Lemma 1 of this paper.

[3] Others, in examining the fundamental properties of non-negative primitive matrices have indirectly obtained bounds for $\gamma(A)$. For example, as pointed out by Wielandt [**2**, p. 647], Frobenius [**1**, p. 463] indirectly obtained the bound $2n^2 - 2n$, while Herstein [**3**, p. 20] indirectly obtained the bound $n^n$ for $\gamma(A)$.

[4] This obviously implies that $A$ is irreducible. See [**1**, p. 463].

PROOF. For $j\in J$, $J=F^{\gamma(A)}(j)\subseteq F^{\gamma(A)}(J)=F[F^{\gamma(A)-1}(J)]\subseteq F(J)\subseteq J$.

LEMMA 2. *If $L$ is a proper subset of $J$, then $F(L)$ contains some element not in $L$.*

PROOF. If not, then $J\supseteq L\supseteq F(L)\supseteq \cdots \supseteq F^{\gamma(A)}(L)=J$ which contradicts $J\neq L$.

COROLLARY. *If $h\leqq n-1$, then $\{j\}\cup F(j)\cup \cdots \cup F^h(j)$ contains at least $h+1$ elements.*

PROOF. This is obviously true for $h=0$. Using mathematical induction, assume it is true for some $0\leqq h\leqq n-1$. Set $L=\{j\}\cup \cdots \cup F^h(j)$, and apply Lemma 2.

We remark that, given $j\in J$, the set of integers $h$ such that $j\in F^h(j)$ is a semigroup. Therefore, properties described below may be easily observed by observing the first few iterates of $A$.

LEMMA 3. *Let $k$ be a non-negative integer, and $j\in J$. For $h\geqq k$, let $j\in F^h(j)$. Then, $F^{n-1+k}(j)=J$.*

PROOF. The corollary above implies that $\{j\}\cup \cdots \cup F^{n-1}(j)=J$. For each $0\leqq h\leqq n-1$, $j\in F^{n-1+k-h}(j)$, and so $F^h(j)\subseteq F^{n-1+k}(j)$. Therefore, $J=\bigcup_{h=0}^{n-1} F^h(j)\subseteq F^{n-1+k}(j)\subseteq J$.

THEOREM 1. *Let $k$ be a non-negative integer. Let there be at least $d>0$ elements $j$ of $J$ such that for $h\geqq k$, the $j$th diagonal element of $A^h$ is positive. Then, $\gamma(A)\leqq 2n-d-1+k$.*

PROOF. The corollary above implies that, for each $j\in J$, there exists $0\leqq h\leqq n-d$ such that $F^h(j)$ contains at least one of the $d$ elements described above. Then,

$$J\supseteq F^{2n-d-1+k}(j) = F^{n-d-h}\{F^{n-1+k}[F^h(j)]\}\supseteq F^{n-d-h}(J) = J.$$

COROLLARY. *Let at least $d>0$ of the diagonal elements of $A$ be positive. Then, $\gamma(A)\leqq 2n-d-1$.*[5]

THEOREM 2. *Let $h$ be a positive integer, and let $A+A^2+\cdots+A^h$ have at least $d>0$ of its diagonal elements positive. Then, $\gamma(A)\leqq n-d+h(n-1)$.*

PROOF. Let $j$ be one of the $d$ elements such that $j\in F^p(j)$ for some $p$, $1\leqq p\leqq h$. Then, if we substitute 0 for $k$, and $F^p$ for $F$, we may apply Lemma 3, and conclude that $F^{(n-1)p}(j)=J$. Choose arbitrarily $j'\in J$. Then, the corollary to Lemma 2 implies that there exists an $l$,

[5] If all the diagonal elements of $A$ are positive, then $d=n$, and the inequality of the corollary reduces to Wielandt's result (2).

$0 \leqq l \leqq n-d$ such that $F^l(j')$ contains at least one of these $d$ elements. Therefore, $J \supseteq F^{n-d+h(n-1)}(j') = F^{n-d-l+(h-p)(n-1)}\left\{F^{p(n-1)}\left[F^l(j')\right]\right\} \supseteq F^{n-d-l+(h-p)(n-1)}(J) = J$, since $n-d-l+(h-p)(n-1) \geqq 0$.

COROLLARY. *Let $A$ be non-negative and positively symmetric in that $a_{i,j} > 0$ if and only if $a_{j,i} > 0$. Then, $\gamma(A) \leqq 2(n-1)$.*

PROOF. $A^2$ has all its diagonal elements positive. Now, apply Theorem 2.

THEOREM 3. $\gamma(A) \leqq n^2 - 2n + 2$.

PROOF. Given $j \in J$, consider the case where $\{j\} \cup \cdots \cup F^{n-2}(j) \neq J$. Then, for $1 \leqq h \leqq n-1$, $F^h(j)$ contains exactly one element not in $\{j\} \cup \cdots \cup F^{h-1}(j)$. Let $p$ be the smallest positive integer such that $F^p(j)$ contains at least two elements. Then, there exists an integer $m < p$ such that $m > 0$ (unless $p = 1$, in which case $m = 0$) and such that $F^m(j) \subseteq F^p(j) = F^{m+(p-m)}(j) \subseteq F^{m+2(p-m)}(j) \subseteq \cdots$. Lemma 2 implies that $F^{m+(n-1)(p-m)}(j) = J$. But $p \leqq n$ implies that

$$m + (n-1)(p-m) = p + (n-2)(p-m) \leqq n^2 - 2n + 2.$$

If $\{j\} \cup \cdots \cup F^{n-2}(j) = J$, then there exists an integer $h$, $0 \leqq h \leqq n-1$, such that $F^0(j) \subseteq F^h(j) \subseteq \cdots \subseteq F^{(n-1)h}(j) = J$. But, $(n-1)h \leqq n^2 - 2n + 1 < n^2 - 2n + 2$. This completes the proof.

Let $A$ and $B$ be two non-negative primitive matrices such that if $A = \|a_{i,j}\|$, and $B = \|b_{i,j}\|$, then $a_{i,j} > 0$ implies that $b_{i,j} > 0$. It is clear that $\gamma(A) \geqq \gamma(B)$. Furthermore, if $B$ has many positive elements for which there are no corresponding positive elements of $A$, then one would expect to have $\gamma(A) > \gamma(B)$. We shall show that when there are many positive off-diagonal elements of a non-negative primitive matrix, some of the preceding inequalities may be improved.

Given a positive integer $j$, $1 \leqq j \leqq n$, define $X(j)$ as the number of elements $a_{i,j}$, $i \neq j$, for which $a_{i,j} > 0$. Then, the corollary to Lemma 2 implies that $X(j) \geqq 1$ whenever $n > 1$, for all $j$. Whenever $X(j) > 1$, we may improve the result of the corollary to Lemma 2 by observing that if $1 \leqq h \leqq n - X(j)$, then $\{j\} \cup F(j) \cup \cdots \cup F^h(j)$ contains at least $h + X(j)$ elements. If we use this result in the proofs of Lemma 3 and Theorem 1, we obtain the following improvements.

LEMMA 4. *Let $k$ and $j$ be as in Lemma 3. Then, $F^{n-X(j)+k}(j) = J$.*

THEOREM 4. *Let $A$ be as in Theorem 1. Let $X_1$ be the minimum of $X(j)$ for the $d$ elements $j \in J$. Let $X_2$ be the minimum of $X(j)$ for the remaining $n-d$ elements $j \in J$. Then,*

$$\gamma(A) \leqq 2n - d - X_1 - \min\left[X_2 - 1;\, n - d\right] + k.$$

COROLLARY. *Let $d>0$ of the diagonal elements of $A$ be positive. Then,*

$$\gamma(A) \leqq 2n - d - X_1 - \min [X_2 - 1; n - d].$$

A similar improvement may also be obtained for Theorem 2.

For any non-negative irreducible matrix, we may define the (irreducible) *order of $A$*, denoted by $\Lambda(A)$, as the smallest positive integer $h$ such that $I+A+A^2+ \cdots +A^h$ is positive, or equivalently, $\{j\} \cup \cdots \cup F^h(j)=J$ for each $j$. By definition of irreducibility, it is clear that $\Lambda(A) \leqq n-1$. If $\Lambda(A)$ is less than $n-1$, and the value of $\Lambda(A)$ is known, many of the preceding inequalities may be improved. We summarize how the order of $A$ may be used to sharpen respectively the results of Lemma 4, Theorem 4, and its corollary above. These results are respectively:

$$F^{\min [n-X(j);\, \Lambda(A)]+k}(j) = J, \tag{3}$$

$$\gamma(A) \leqq \min [n - X_1; \Lambda(A)] + \min \{n - d - \min [X_2 - 1; n - d]; \Lambda(A)\} + k, \tag{4}$$

$$\gamma(A) \leqq \min[n - X_1; \Lambda(A)] + \min \{n - d - \min [X_2 - 1; n - d]; \Lambda(A)\}. \tag{5}$$

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