ON TORSION-FREE ABELIAN GROUPS AND LIE ALGEBRAS

RICHARD BLOCK

It is known that many of the classes of simple Lie algebras of prime characteristic of nonclassical type have simple infinite-dimensional analogues of characteristic zero (see, for example, [4, p. 518]). We consider here analogues of those algebras which are defined by a modification of the definition of a group algebra. Thus we consider analogues of the Zassenhaus algebras as generalized by Albert and Frank in [1].

The algebras considered are defined as follows. Let G be a nonzero abelian group, F a field, g an additive mapping from G to F, and f an alternate biadditive mapping from $G\times G$ to F. We index a basis of an algebra over F by G, denoting by u_α the basis element corresponding to α in G, and define multiplication by

$$(1) \qquad u_\alpha u_\beta = \{f(\alpha, \beta) + g(\alpha - \beta)\} u_{\alpha+\beta}.$$

We designate this algebra by $L(G, g, f)$. We shall determine necessary and sufficient conditions on f and g for the algebra $L(G, g, f)$ to be a simple Lie algebra. We shall then consider the case in which $L(G, g, f)$ is a simple Lie algebra of characteristic zero. This will be seen to imply that G is torsion-free. The derivations and locally algebraic derivations of $L(G, g, f)$ will be determined in this case. Using these, we shall show that any one of these simple Lie algebras $L(G, g, f)$ of characteristic zero determines the group G up to isomorphism and determines the mappings g and f up to a scalar multiple.

Our proof of the simplicity of $L(G, g, f)$ and determination of the derivations of $L(G, g, f)$ are also valid when F has prime characteristic p and G is an elementary abelian p-group. However in that case our method for showing that $L(G, g, f)$ essentially determines G, g and f cannot be used—indeed, Ree showed in [4] that all Zassenhaus algebras of dimension p^n over F are isomorphic.

When the torsion-free abelian group G has rank one, the simple algebra $L(G, g, f)$ over F, of characteristic zero, is isomorphic to the algebra of derivations of the group algebra of G over F. Thus the group algebra of a torsion-free abelian group of rank one determines the group. However this is a special case of a result that follows from Higman's determination of the units of group algebras in [2].

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2. **Simplicity.** It was shown by Albert and Frank in [**1**, p. 132] that when $g(\alpha)=0$ for all α in G then the algebra $L(G, g, f)$ is a Lie algebra and contains the one-dimensional ideal spanned by u_0, and that when g does not vanish identically then $L(G, g, f)$ is a Lie algebra if and only if there is an (additive) mapping h on G to F such that

$$f(\alpha, \beta) = g(\alpha)h(\beta) - g(\beta)h(\alpha) \tag{2}$$

for all α and β in G.

Now suppose that g does not vanish identically and that an h is given such that (2) holds. We seek to determine conditions under which $L(G, g, f)$ will be simple. If $g(\gamma)=h(\gamma)=0$ for some nonzero γ then in case γ has finite order q, all elements of the form

$$\sum_{i=0}^{q-1} u_{\alpha+i\gamma}$$

span a proper ideal, while if γ has infinite order, all elements of the form $u_\alpha+u_{\alpha+\gamma}$ span a proper ideal. Also if there is an element δ in G such that $g(\delta)=0$ and $h(\delta)=2$ (in particular for characteristic two, if $\delta=0$) then all u_α with $\alpha\neq-\delta$ span a proper ideal of $L(G, g, f)$.

We therefore suppose that there is no nonzero δ in G such that $g(\delta)=0$ and $h(\delta)=0$ or 2, and that the characteristic is not two. Thus there is no δ in G such that $g(\delta)=0$ and $h(\delta)=\pm 1$. Also, by (2), either the kernel of g is zero or f is nonsingular.

The *length* $\lambda(x)$ of an element $x=\sum_\beta a_\beta u_\beta$ of $L(G, g, f)$ denotes the number of nonzero coefficients a_β in x, and β is said to be *x-admissible* if $a_\beta\neq 0$.

Now, under the above conditions, let M be a nonzero ideal of $L(G, g, f)$, x a nonzero element of M of minimal length, and suppose $\lambda(x)>1$. If $g(\alpha)=0$ for some x-admissible α, then with γ such that $g(\gamma)\neq 0$, we have $\lambda(xu_\gamma)\leqq\lambda(x), f(\alpha,\gamma)+g(\alpha-\gamma)=-g(\gamma)[h(\alpha)+1]\neq 0$ and $\alpha+\gamma$ is (xu_γ)-admissible. Hence we may assume that $g(\beta)\neq 0$ for some x-admissible β. Now if $g(\alpha)=0$ for some x-admissible α then $xu_\alpha\neq 0$ since $f(\beta, \alpha)+g(\beta-\alpha)=g(\beta)[h(\alpha)+1]\neq 0$, but $\lambda(xu_\alpha)<\lambda(x)$, a contradiction. Hence for every x-admissible β, $g(\beta)\neq 0$. Take an x-admissible α and let $y=xu_{-\alpha}$. Then 0 is y-admissible since $f(\alpha, -\alpha)+g(\alpha)-g(-\alpha)=2g(\alpha)\neq 0$, so $y\neq 0$ and $\lambda(y)=\lambda(x)$. Thus $\beta-\alpha$ is y-admissible for every x-admissible β, and since 0 is y-admissible, $g(\beta-\alpha)=0$ for every x-admissible β. Now if α and β are x-admissible and $\alpha\neq\beta$, then since $\lambda(xu_\beta)<\lambda(x)$, we have $xu_\beta=0$, $0=f(\alpha, \beta)+g(\alpha-\beta)=g(\alpha)[h(\beta-\alpha)]$, so $g(\alpha-\beta)=h(\alpha-\beta)=0$ and $\alpha=\beta$, a contradiction.

Hence $\lambda(x)=1$ and M contains some u_α. If $g(\alpha)=0$ then for β such that $g(\beta)\neq 0$, M contains $u_\alpha u_\beta = -g(\beta)[h(\alpha)+1]u_{\alpha+\beta}\neq 0$, so M contains a u_γ with $g(\gamma)\neq 0$. Now if β is such that $g(\beta)=0$ then M contains u_β, since $u_\gamma u_{-\gamma+\beta}=g(\gamma)[h(\beta)+2]u_\beta\neq 0$. In particular M contains u_0 and so also any u_α with $g(\alpha)\neq 0$, since $u_0u_\alpha=-g(\alpha)u_\alpha\neq 0$. Therefore $M=L$, and we have proved the following theorem.

THEOREM 1. *The algebra $L(G, g, f)$ defined by* (1) *is a simple Lie algebra if and only if the characteristic is not two, g does not vanish identically and there is an additive mapping h from G to F for which* (2) *holds and for which there is no nonzero δ in G with $g(\delta)=0$ and $h(\delta)=0$ or* 2.

From the conditions of this theorem on the mappings g and h, the following result may be proved easily.

COROLLARY 1. *Suppose that F is a field of characteristic* 0, *of degree d* (*finite or infinite*) *over the rationals. Then if $L(G, g, f)$ is a simple Lie algebra over F, the group G must be torsion-free. Let G be a given torsion-free abelian group and let r be the rank* (*the maximum number of linearly independent elements*) *of G. Then there exists a simple Lie algebra $L(G, g, f)$ over F if and only if $2d\geqq r$, when G is not divisible, or $2d\geqq r+1$, when G is divisible.*

Similar statements hold when F has prime characteristic p. In that case G must be an elementary p-group in order for $L(G, g, f)$ to be a simple Lie algebra. The Lie algebras $L(G, g, f)$ were shown to be simple when g is an isomorphism by Albert and Frank in [**1**, p. 138]. In the finite dimensional case the simple Lie algebras $L(G, g, f)$ may be shown to be the same up to isomorphism as the p^n-dimensional simple Lie algebras considered by Jennings and Ree in [**3**].

3. **The algebra of derivations of** $L(G, g, f)$. We shall henceforth assume that any algebra $L(G, g, f)$ considered is a simple Lie algebra, and in particular that a mapping h satisfying the conditions of Theorem 1 is given.

Now suppose that D is a derivation of $L(G, g, f)$ and let $c(\alpha, \gamma)$ be the coefficient of $u_{\alpha+\gamma}$ in $u_\alpha D$, that is,

$$D\colon u_\alpha \to \sum_{\gamma\in G} c(\alpha, \gamma)u_{\alpha+\gamma} = \sum_{\gamma\in G} c(\alpha, -\alpha+\gamma)u_\gamma,$$

the sums being finite of course. Since D is a derivation, $(u_\gamma u_\epsilon)D=(u_\gamma D)u_\epsilon+u_\gamma(u_\epsilon D)$, that is, with $\phi(\alpha,\beta)$ denoting $f(\alpha,\beta)+g(\alpha-\beta)$,

$$\sum_{\zeta\in G}\{\phi(\gamma,\epsilon)c(\gamma+\epsilon, -\gamma-\epsilon+\zeta) - \phi(\zeta-\epsilon,\epsilon)c(\gamma, -\gamma-\epsilon+\zeta) - \phi(\gamma, -\gamma+\zeta)c(\epsilon, -\gamma-\epsilon+\zeta)\}u_\zeta = 0$$

for all γ and ϵ in G. Taking $\zeta=\gamma+\epsilon+\theta$ this gives

$$(3)\quad \begin{aligned}[f(\gamma,\epsilon)+g(\gamma-\epsilon)]c(\gamma+\epsilon,\theta) = {} & [f(\gamma+\theta,\epsilon)+g(\gamma-\epsilon+\theta)]c(\gamma,\theta)\\ & + [f(\gamma,\epsilon+\theta)+g(\gamma-\epsilon-\theta)]c(\epsilon,\theta)\end{aligned}$$

for all γ, ϵ and θ in G.

LEMMA 1. *If $\theta\neq 0$ then*

$$(4)\qquad [f(\beta,\theta)+g(\beta-\theta)]c(\alpha,\theta) = [f(\alpha,\theta)+g(\alpha-\theta)]c(\beta,\theta)$$

for any α and β in G.

In this proof we denote $c(\gamma,\theta)$ by c_γ for any γ. The proof is divided into two cases.

CASE I. $g(\theta)\neq 0$. Taking $\gamma=\alpha$ and $\epsilon=0$ in (3), we get $0=g(\theta)c_\alpha+[f(\alpha,\theta)+g(\alpha-\theta)]c_0$, i.e.,

$$c_\alpha = [f(\alpha,\theta)+g(\alpha-\theta)][-g(\theta)]^{-1}c_0$$

which, together with the similar result for β, gives (4).

CASE II. $g(\theta)=0$, $\theta\neq 0$. If $g(\alpha)=g(\beta)=0$ then both sides of (4) vanish, so we may assume that, say, $g(\alpha)\neq 0$. With $\gamma=\alpha$ and $\epsilon=0$, (3) gives $g(\alpha)[h(\theta)+1]c_0=0$, i.e., $c_0=0$. Now let ζ and η be any nonzero elements of G such that $g(\zeta)\neq 0$ and $g(\eta)=0$. Then (3) with $\gamma=\zeta$ and $\epsilon=-\zeta$ gives $0=g(\zeta)[h(\theta)+2]\{c_\zeta+c_{-\zeta}\}$, i.e.,

$$c_{-\zeta} = -c_\zeta.$$

Also by (3), $g(\zeta)[h(\eta)+1]c_{\zeta+\eta}=g(\zeta)[h(\eta)+1]c_\zeta+g(\zeta)[h(\eta+\theta)+1]c_\eta$, i.e.,

$$c_{\zeta+\eta} = c_\zeta + [h(\eta)+1]^{-1}[h(\eta+\theta)+1]c_\eta,$$

while with $\gamma=-\zeta$ and $\epsilon=\zeta+\eta$, (3) gives

$$-g(\zeta)[h(\eta)+2]c_\eta = -g(\zeta)[h(\eta+\theta)+2]\{c_{-\zeta}+c_{\zeta+\eta}\}$$

i.e.,

$$c_{\zeta+n} = -c_{-\zeta} + [h(\eta+\theta)+2]^{-1}[h(\eta)+2]c_\eta.$$

Hence $[h(\eta+\theta)+2][h(\eta+\theta)+1]c_\eta=[h(\eta)+1][h(\eta)+2]c_\eta$, so that $h(\theta)[h(2\eta+\theta)+3]c_\eta=0$. Thus $c_\eta=0$ and $c_{\zeta+\eta}=c_\zeta$ if $h(2\eta+\theta)\neq -3$ while if $h(2\eta+\theta)=-3$ then, since $h(4\eta+\theta)\neq -3$ and $h(-2\eta+\theta)\neq -3$, we have $c_{\zeta+\eta}=c_{\zeta-\eta+2\eta}=c_{\zeta-\eta}=c_\zeta$, and again $c_\eta=0$. It follows that

$$c_{\alpha+\beta+\theta} = c_{\alpha+\beta}, \qquad c_{\beta+\theta} = c_\beta.$$

Now with $\gamma=\alpha$ and $\epsilon=\beta+\theta$ in (3) we have

$$[f(\alpha, \beta + \theta) + g(\alpha - \beta)]c_{\alpha+\beta}$$
$$= [f(\alpha + \theta, \beta + \theta) + g(\alpha - \beta)]c_\alpha + [f(\alpha, \beta + 2\theta) + g(\alpha - \beta)]c_\beta.$$

Subtracting from this (3) with $\gamma = \alpha$ and $\epsilon = \beta$ we get $f(\alpha, \theta)c_{\alpha+\beta} = f(\alpha, \theta)\{c_\alpha + c_\beta\}$, so that $c_{\alpha+\beta} = c_\alpha + c_\beta$. Now (3) with $\gamma = \alpha$ and $\epsilon = \beta$ gives

$$f(\beta, \theta)c_\alpha = f(\alpha, \theta)c_\beta.$$

Multiplying both sides of this by $[h(\theta)]^{-1}[h(\theta)+1]$, we get (4) for this case also, which proves the lemma.

LEMMA 2. *The derivation D differs by an inner derivation D' from a derivation for which the coefficients $c(\alpha, \theta)$ vanish for all nonzero θ.*

Indeed for any nonzero θ we may take an α such that $f(\alpha, \theta) + g(\alpha - \theta) \neq 0$ and set $k_\theta = [f(\alpha, \theta) + g(\alpha - \theta)]^{-1}c(\alpha, \theta)$. By Lemma 1, k_θ is well defined. Thus if $g(\theta) \neq 0$ then $k_\theta = -[g(\theta)]^{-1}c(0, \theta)$, and since $c(0, \theta) \neq 0$ for only finitely many θ, there are only finitely many θ such that $g(\theta) \neq 0$ and $k_\theta \neq 0$. Similarly, taking α such that $g(\alpha) \neq 0$, we find that there are also only finitely many θ such that $g(\theta) = 0$ and $k_\theta \neq 0$. Therefore we may consider the right multiplication by $\sum_{\theta \neq 0} k_\theta u_\theta$. Taking this to be D', and noting that by Lemma 1 if $f(\alpha, \theta) + g(\alpha - \theta) = 0$ then $c(\alpha, \theta) = 0$ (for nonzero θ), we see that the lemma holds.

Now let d be any additive function of G to F. Then it is easy to see that the linear transformation D_d determined by the mapping of the basis elements

$$u_\alpha \to d(\alpha)u_\alpha$$

is a derivation of $L(G, g, f)$.

THEOREM 2. *The algebra of derivations of the simple Lie algebra $L(G, g, f)$ is spanned by the inner derivations together with all the derivations D_d.*

What remains to be proved is that

$$c_{\alpha+\beta} = c_\alpha + c_\beta \tag{5}$$

for any α and β, where, for any γ, c_γ denotes $c(\gamma, 0)$. If $f(\alpha, \beta) + g(\alpha - \beta) \neq 0$, (5) follows directly from (3) with $\theta = 0$. But if $f(\alpha, \beta) + g(\alpha - \beta) = 0$ then we may pick a γ such that the expressions

$$f(\alpha + \beta, \gamma) + g(\alpha + \beta - \gamma), \qquad f(\alpha, \beta + \gamma) + g(\alpha - \beta - \gamma),$$
$$f(\beta, \gamma) + g(\beta - \gamma)$$

are all nonzero, for if $g(\alpha)=0$ (so that $g(\beta)=0$ also) then any γ for which $g(\gamma)\neq 0$ will do, while if $g(\alpha)\neq 0$ we may take $\gamma=2\beta$ unless $\alpha=\beta$. When $g(\alpha)\neq 0$ and $\alpha=\beta$ (and the characteristic is not 3) we may take $\gamma=-\alpha$. Now with such a γ we have

$$c_{\alpha+\beta}+c_\gamma=c_{\alpha+\beta+\gamma}=c_\alpha+c_{\beta+\gamma}=c_\alpha+c_\beta+c_\gamma$$

(when the characteristic is 3 and $\alpha=\beta$ one argues that $c_{2\alpha}=c_{-\alpha}=-c_\alpha=2c_\alpha$). Thus (5) holds and the theorem is proved.

Noting that the derivation sending u_α to $g(\alpha)u_\alpha$ is inner, we obtain the following result.

COROLLARY 2. *If G has finite rank n then the algebra of outer derivations of $L(G, g, f)$ is an abelian Lie algebra of dimension $n-1$.*

4. Criteria for isomorphism. Henceforth we shall restrict our consideration to simple Lie algebras $L(G, g, f)$ of characteristic zero, so that G must be torsion-free.

A derivation D of an algebra L is locally algebraic if and only if it is true that for every x in L the set $\{xD^i: i=1, 2, \cdots\}$ lies in a finite-dimensional subspace (depending on x) of L.

LEMMA 3. *The only locally algebraic derivations of $L(G, g, f)$ are the derivations D_d.*

The derivations D_d are obviously locally algebraic. Now suppose that D is a locally algebraic derivation. By Theorem 2, $D=R_y+D_d$, where R_y is the right multiplication by $y=\sum_\gamma a_\gamma u_\gamma$, for some y and d. Suppose that some nonzero γ is y-admissible. We may simply order G in such a way that this $\gamma>0$. Call u_ϵ the leading term in an element z of $L(G, g, f)$ if ϵ is the greatest z-admissible element of G, and let u_α be the leading term in y. Thus $\alpha>0$. We shall find a β such that the leading term in $(u_\beta)D^i$ is $u_{\beta+i\alpha}$, contradicting the assumption that D is locally algebraic. Indeed if $g(\alpha)=0$ we may take β to be any positive element of G with $g(\beta)\neq 0$, since then the coefficient of $u_{\beta+i\alpha}$ in $(u_\beta)D^i$ is $[g(\beta)]^i[h(\alpha)+1]^i\neq 0$, while if $g(\alpha)\neq 0$ we may take β to be 2α. Thus y must be a scalar multiple of u_0, and the lemma is proved.

Since for any distinct elements α and β there is an additive mapping d of G to F such that $d(\alpha)\neq d(\beta)$, we have the following result.

LEMMA 4. *The only elements of $L(G, g, f)$ which are characteristic vectors for all locally algebraic derivations are the scalar multiples of all the elements u_α.*

Now suppose that σ is an isomorphism of one algebra $L(G, g, f)$ onto another, $L(G', g', f')$. We shall determine the relations between

G, g and f on the one hand, and G', g' and f' on the other. It follows from Lemma 4 that for every α in G there is an element α^σ in G' such that

$$(u_\alpha)\sigma = cl_\alpha u_{\alpha^\sigma} \tag{6}$$

where l_α is a nonzero scalar (depending on α and σ) and c is a fixed scalar chosen so that $l_0=1$. The induced mapping $\sigma\colon \alpha\to\alpha^\sigma$ of G into G' is one-to-one and onto. Since $(u_\gamma u_\epsilon)\sigma=[(u_\gamma)\sigma][(u_\epsilon)\sigma]$,

$$\begin{aligned} cl_{\gamma+\epsilon}[f(\gamma,\epsilon)+g(\gamma-\epsilon)]u_{(\gamma+\epsilon)^\sigma} \\ = c^2 l_\gamma l_\epsilon[f'(\gamma^\sigma,\epsilon^\sigma)+g'(\gamma^\sigma-\epsilon^\sigma)]u_{\gamma^\sigma+\epsilon^\sigma} \end{aligned} \tag{7}$$

for all γ and ϵ in G. Hence if $f(\gamma,\epsilon)+g(\gamma-\epsilon)\neq 0$ then $(\gamma+\epsilon)^\sigma=\gamma^\sigma+\epsilon^\sigma$, so that, exactly as in the final part of the proof of Theorem 2, σ is always additive and therefore is an isomorphism of G onto G'.

Taking $\gamma=\alpha$ and $\epsilon=0$ in (7) we get

$$g'(\alpha^\sigma) = c^{-1}g(\alpha) \tag{8}$$

for any α in G. Now taking $\epsilon=-\gamma$ in (7), we have $l_{-\gamma}=l_\gamma^{-1}$ for any γ, and taking $\gamma=2\zeta$ and $\epsilon=-\zeta$ we have $l_\zeta=l_{2\zeta}l_{-\zeta}$, i.e., $l_{2\zeta}=l_\zeta^2$ for any ζ in G. If $f(\alpha,\beta)+g(\alpha-\beta)=0$ then, by (7) and (8), $cf'(\alpha^\sigma,\beta^\sigma)+g(\alpha-\beta)=0$ and

$$f'(\alpha^\sigma,\beta^\sigma) = c^{-1}f(\alpha,\beta). \tag{9}$$

Similarly, if $f(\alpha,\beta)-g(\alpha-\beta)=0$ or $4f(\alpha,\beta)+2g(\alpha-\beta)=0$ then by taking $\gamma=-\alpha$ and $\epsilon=-\beta$, or $\gamma=2\alpha$ and $\epsilon=2\beta$, in (7), we have (9) again. Now suppose that the expressions $f(\alpha,\beta)+g(\alpha-\beta)$, $f(\alpha,\beta)-g(\alpha-\beta)$ and $4f(\alpha,\beta)+2g(\alpha-\beta)$ are nonzero. By (7) and (8) we have

$$l_{-\alpha-\beta} = l_{-\alpha}l_{-\beta}[cf'(\alpha^\sigma,\beta^\sigma)-g(\alpha-\beta)][f(\alpha,\beta)-g(\alpha-\beta)]^{-1}$$

while on the other hand

$$l_{-\alpha-\beta} = (l_{\alpha+\beta})^{-1} = l_\alpha^{-1}l_\beta^{-1}[cf'(\alpha^\sigma,\beta^\sigma)+g(\alpha-\beta)]^{-1}[f(\alpha,\beta)+g(\alpha-\beta)].$$

Hence $[cf'(\alpha^\sigma,\beta^\sigma)]^2=[f(\alpha,\beta)]^2$. Now similarly by expanding $l_{2\alpha+2\beta}$ in two different ways one may see that (9) always holds. It then follows that $l_{\alpha+\beta}=l_\alpha l_\beta$ for any α and β in G. We have thus proved one direction of the following theorem. The converse is clear from (7).

THEOREM 3. *A linear mapping $\sigma\colon L(G,g,f)\to L(G',g',f')$ of one of the simple Lie algebras $L(G,g,f)$ of characteristic zero onto another is an isomorphism if and only if there is an induced isomorphism $\sigma\colon\alpha\to\alpha^\sigma$ of G onto G', a nonzero scalar c and a homomorphism $l\colon\alpha\to l_\alpha$ of G into the multiplicative group F^* of the base field, such that* (6), (8) *and* (9) *hold for all α and β in G.*

This result in particular applies to automorphisms of $L(G, g, f)$. Thus if G has rank one then the automorphism group of $L(G, g, f)$ is a semidirect product of a normal subgroup A and a subgroup B, where A is isomorphic to the group of homomorphisms of G into F^* and B is isomorphic to the group of automorphisms of G. Also if G is a free abelian group on n generators $\alpha_1, \cdots, \alpha_n$ and the elements $g(\alpha_i)g(\alpha_j)$ $(1 \leqq i \leqq j \leqq n)$ of F are linearly independent over the rationals, then the automorphism group of $L(G, g, f)$ is again a semidirect product of groups A and B, where A is an n-fold direct product of F^*, and B has order 2 or 1 according to whether f vanishes identically or not.

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Indiana University