THE DISTRIBUTION OF \( a^-\) POINTS OF AN ENTIRE FUNCTION

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1. Let \( f(z) \) be an entire function of order \( \rho \) (\( 0 < \rho < \infty \)) and lower order \( \lambda \) (\( 0 \leq \lambda < \infty \)). It is known that corresponding to every entire function of finite nonzero order there exists a function \( \rho(r) \) called its proximate order having the following properties:

(1.1) \( \rho(r) \) is real, continuous and piecewise differentiable,
(1.2) \( \rho(r) \to \rho \) as \( r \to \infty \),
(1.3) \( r\rho'(r) \log r \to 0 \) as \( r \to \infty \),

\[
\log M(r, f) \leq r^{\rho(r)} \quad \text{for} \quad r \geq r_0 \\
= r^{\rho(r)} \quad \text{for a sequence of values of} \quad r.
\]

2. S. M. Shah [2] has proved the existence of a function \( \lambda(r) \) for an entire function of lower order \( \lambda \) (\( 0 \leq \lambda < \infty \)) analogous to \( \rho(r) \), having the following properties:

(2.1) \( \lambda(r) \) is a non-negative, continuous function of \( r \) for \( r \geq r_0 \).
(2.2) \( \lambda(r) \) is differentiable except at isolated points at which \( \lambda'(r-0) \) and \( \lambda'(r+0) \) exist.
(2.3) \( \lambda(r) \to \lambda \) as \( r \to \infty \).
(2.4) \( r\lambda'(r) \log r \to 0 \) as \( r \to \infty \).

\[
\log M(r, f) \geq r^{\lambda(r)} \quad \text{for} \quad r \geq r_0 \\
= r^{\lambda(r)} \quad \text{for a sequence of values of} \quad r.
\]

3. In this note we prove a number of results applying the properties of \( \lambda(r) \) and \( \rho(r) \). In what follows we shall take \( 0 < \lambda < \infty \). From properties (2.1)–(2.5) of \( \lambda(r) \) we can easily deduce that \( r^{\lambda(r)} \) is an increasing function of \( r(r \geq r_0) \), for

\[
\frac{d}{dr} (r^{\lambda(r)}) = (\lambda(r) + \lambda'(r))r^{\lambda(r)-1} > 0 \quad \text{for} \quad r \geq r_0.
\]

With the usual notations of \( \log M(r, f) \), \( n(r, a) \) and \( N(r, a) \) we prove the following theorems:

**Theorem 1.** If

\[
\limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} < \infty
\]

and

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(3.2) \[ \frac{N(r, a)}{r^{\lambda(r)}} \to 0 \text{ as } r \to \infty, \]

then for \[ x \neq a \]

(i) \[ 0 < \liminf_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq 1, \]

(ii) \[ \left( \frac{h - 1}{h + 1} \right) \frac{1}{h^\lambda} \leq \limsup_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq (1/\lambda) \limsup_{r \to \infty} \frac{n(r, x)}{r^{\lambda(r)}} < \infty \]

where \( h = \frac{(1 + (1 + \lambda^2)^{1/2})}{\lambda}. \)

**Theorem 2.**

(i) \[ \liminf_{r \to \infty} \frac{n(r)}{r^{\lambda(r)}} \leq \lambda, \]

(ii) \[ \liminf_{r \to \infty} \frac{n(r)}{r^{\rho(r)}} \leq \rho. \]

**Theorem 3.**

(i) If \[ \lim_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}} \text{ exists, then } \lim_{r \to \infty} \frac{n(r, x)}{r^{\lambda(r)}} = \lambda \lim_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}}. \]

(ii) If \[ \lim_{r \to \infty} \frac{N(r, x)}{r^{\rho(r)}} \text{ exists, then } \lim_{r \to \infty} \frac{n(r, x)}{r^{\rho(r)}} = \rho \lim_{r \to \infty} \frac{N(r, x)}{r^{\rho(r)}}. \]

**Theorem 4.** If \( f(z) \) be an entire function of finite nonzero order for which \[ \frac{n(r, a)}{\log M(r, f)} \to 0 \quad \text{as } r \to \infty, \]

then \[ \liminf_{r \to \infty} \frac{N(r, a)}{\log M(r, f)} = 0. \]

We observe that the above theorem does not hold if \( \rho = 0 \). For instance consider \[ f(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{e^n} \right). \]

Then \( f(z) \) is an entire function of zero order for which
\[ n(r, 0) \sim \log r, \]
\[ N(r, 0) \sim 1/2(\log r)^2, \]
\[ \log M(r, f) \sim N(r, 0); \]
hence
\[ \frac{n(r, 0)}{\log M(r, f)} \to 0, \quad \text{but} \quad \frac{N(r, 0)}{\log M(r, f)} \to 1. \]

As another example we can take any polynomial \( P(z) \) then
\[ \frac{n(r, 0)}{\log M(r, P)} \to 0, \quad \text{but} \quad \frac{N(r, 0)}{\log M(r, P)} \to K \quad (K > 0). \]

**Theorem 5.** Let \( f(z) \) be an entire function of order \( \rho \) (\( 0 < \rho < \infty \)) such that

\begin{enumerate}
  \item \[
  \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\rho(r)}} > 0,
  \]
  \item \[
  \lim_{r \to \infty} \frac{N(r, a)}{r^{\rho(r)}} = 0,
  \]
\end{enumerate}

then
\[ 0 < \liminf_{r \to \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq \limsup_{r \to \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq 1 \]
for all \( x \neq a \).

In the above theorem Condition (1) namely
\[ \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\rho(r)}} > 0 \]
is essential, because there exist entire functions \( f(z) \) for which
\[ \liminf_{r \to \infty} \frac{N(r, a)}{r^{\rho(r)}} = 0 \] for \( \nu = 1, 2, \ldots, k \).

For instance see S. M. Shah and S. K. Singh [4, Theorem I(ii)]. There
\[ \lambda_1(-a) = \liminf_{r \to \infty} \frac{\log n(r, f + a)}{\log r} < \lambda(f) = \liminf_{r \to \infty} \frac{\log \log M(r)}{\log r} \]
\((\nu = 1, 2, \ldots, k)\)
and since
\[ \liminf_{r \to \infty} \frac{\log n(r, f + a)}{\log r} = \liminf_{r \to \infty} \frac{\log N(r, f + a)}{\log r} \]
so \( N(r, f + a) < r^{\lambda_1(-a) + \epsilon} \) for a sequence of values of \( r \), also \( \log M(r, f) > r^{\lambda - \epsilon} \) for all \( r \geq r_0 \), so \( \liminf_{r \to \infty} N(r, f + a) / \log M(r, f) = 0 \); and hence
a fortiori \( \liminf_{r \to \infty} N(r, f + a_v) / r^\rho = 0 \) \((v = 1, 2, \ldots, k)\).

4. **Lemma 1.** \((hr)^{\lambda(hr)} \sim h^\lambda r^{\lambda(r)}\).

**Lemma 2.** \(\int_r^{hr} \lambda(t) \, dt \sim r^{\lambda(r)} / \lambda\).

**Proof of Lemma 1.** It is sufficient to prove that \(r^{\lambda(hr)} \sim r^{\lambda(r)}\).

Now
\[
\lambda(hr) - \lambda(r) = \int_r^{hr} \lambda'(t) \, dt = o\left( \int_r^{hr} \frac{dt}{t \log t} \right) = o\left( \frac{1}{\log r} \right).
\]

Hence
\[
r^{\lambda(hr) - \lambda(r)} \to 1.
\]

Proof of Lemma 2 is similar to [1, Lemma 4, p. 58].

**Proof of Theorem 1(i).** From (2.5) we have
\[
\liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} = 1.
\]

Hence the right hand inequality is obvious as \(N(r, x) \leq \log M(r, f)\).

Also clearly \(N(r, x) > N(r, a), (x \neq a)\) for if \(N(r, x) \leq N(r, a)\), then from Nevanlinna’s second theorem
\[
T(r, f) < N(r, a) + N(r, x) + O(\log r) \\
\leq 2N(r, a) + O(\log r),
\]
\[
\frac{T(r, f)}{r^{\lambda(r)}} \leq \frac{2N(r, a)}{r^{\lambda(r)}} + o(1).
\]

Hence, \(T(r, f) / r^{\lambda(r)} \to 0\) and as \(T(r, f) \leq \log M(r, f) \leq 3T(2r, f)\); so \(\log M(r, f) / r^{\lambda(r)} \to 0\) as \(r \to \infty\); this contradicts (4.1).

Hence, appealing to Nevanlinna’s second theorem again we have
\[
T(r, f) < 2N(r, x) + O(\log r),
\]
\[
\frac{T(r, f)}{r^{\lambda(r)}} < \frac{2N(r, x)}{r^{\lambda(r)}} + o(1).
\]

Hence, \(2N(2r, x) / (2r)^{\lambda(2r)} > T(2r, f) / (2r)^{\lambda(2r)} > A \log M(r, f) / r^{\lambda(r)}\)

and
\[
\liminf_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}} \geq A \liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} = A > 0.
\]

(ii) Now, \(\limsup_{r \to \infty} N(r, a) / r^{\lambda(r)} + \limsup_{r \to \infty} N(r, x) / r^{\lambda(r)} \geq \limsup_{r \to \infty} T(r, f) / r^{\lambda(r)}\) and as \(\limsup_{r \to \infty} N(r, a) / r^{\lambda(r)} = 0\), so
\[
\limsup_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}} = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\lambda(r)}}.
\]

Also, \(T(hr, f) \geq (h-1)/(h+1) \log M(r, f), (h > 1)\) so,
\[
\limsup_{r \to \infty} \frac{T(hr, f)}{(hr)^{\lambda(hr)}} \geq \frac{h-1}{h+1} \frac{1}{h^\lambda} \limsup_{r \to \infty} \log M(r, f) \geq \frac{h-1}{h+1} \frac{1}{h^\lambda},
\]

since, \(\limsup_{r \to \infty} \log M(r, f)/r^{\lambda(r)} \geq 1 \) by (4.1).

Now choosing the best possible value of \(h\) which is
\[
h = (1 + (1 + \lambda^2)^{1/2})/\lambda
\]
we have
\[
\frac{h-1}{h+1} \frac{1}{h^\lambda} \leq \limsup_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}}.
\]

Let now \(\limsup_{r \to \infty} n(r, x)/r^{\lambda(r)} = H\), then,
\[
N(r, x) < \int_{r_0}^r (H + \epsilon) t^{\lambda(t) - 1} dt \sim \frac{H + \epsilon}{\lambda} r^{\lambda(r)}.
\]

Hence,
\[
\limsup_{r \to \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \frac{H}{\lambda} = \frac{1}{\lambda} \limsup_{r \to \infty} \frac{n(r, x)}{r^{\lambda(r)}}.
\]

Further from Jensen's theorem we have
\[
n(r, x) \log 2 < \int_r^{2r} \frac{n(t, x)}{t} dt < \int_0^{2r} \frac{n(t, x)}{t} dt < \log M(2r, f).
\]

Hence, \(n(r, x) \log 2 / r^{\lambda(r)} \leq [(\log M(2r, f)/(2r)^{\lambda(2r)})((2r)^{\lambda(2r)}/r^{\lambda(r)})]\); so,
\[
\limsup_{r \to \infty} \frac{n(r, x)}{r^{\lambda(r)}} \leq A \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} < \infty.
\]

Proof of Theorem 2(i). Let \(\inf_{r \to \infty} n(r)/r^{\lambda(r)} = H\), then
\[
n(r) > (H - \epsilon)r^{\lambda(r)} \quad \text{for } r \geq r_0;
\]
so,
\[
N(r) > \int_{r_0}^r (H - \epsilon) t^{\lambda(t) - 1} dt = \frac{(H - \epsilon)}{\lambda} \log M(r, f)
\]
for a sequence of values of \( r \).

Hence, \( \limsup_{r \to \infty} N(r)/\log M(r, f) \geq H/\lambda \) and so

\[
H/\lambda \leq \limsup_{r \to \infty} N(r)/\log M(r, f) \leq 1.
\]

Hence, \( H \leq \lambda \).

The proof of (ii) is similar.

**Proof of Theorem 3(i).** Let \( \lim_{r \to \infty} N(r, x)/r^\lambda(r) = M \). Set \( N(r, x) = N(r) \), then

\[
(M - \epsilon)r^\lambda(r) < N(r) < (M + \epsilon)r^\lambda(r).
\]

\[
\int_r^{r(1+\alpha)} \frac{n(l)}{l} \, dl = N(r + r\alpha) - N(r) < (M + \epsilon)(r + r\alpha)^\lambda(r + r\alpha) - (M - \epsilon)r^\lambda(r)
\]

\[
= r^\lambda(r) \left\{ (M + \epsilon)(1 + \alpha)^\lambda - (M - \epsilon) \right\}
\]

\[
= r^\lambda(r) \left\{ (M + \epsilon) \left(1 + \lambda\alpha + \frac{\lambda(\lambda - 1)}{2!} \alpha^2 + \cdots \right) - M + \epsilon \right\}
\]

\[
= r^\lambda(r) \left\{ (M\lambda\alpha + \frac{M\lambda(\lambda - 1)}{2!} \alpha^2 + \cdots) + 2\epsilon + \epsilon\lambda \alpha \right\}.
\]

Hence

\[
\frac{n(r)}{r^\lambda(r)} \frac{\alpha}{1 + \alpha} < \int_r^{r(1+\alpha)} \frac{n(l)}{l} \, dl
\]

\[
< \left\{ M\lambda\alpha + 2\epsilon + \epsilon\lambda + \frac{M\lambda(\lambda - 1)\alpha^2}{2!} + \cdots \right\},
\]

\[
\frac{n(r)}{r^\lambda(r)} < (1 + \alpha) \left\{ M\lambda + \frac{2\epsilon}{\alpha} + \epsilon\lambda + \frac{M\lambda(\lambda - 1)}{2!} \alpha + \cdots \right\}.
\]

Setting first \( \alpha = \epsilon^{1/2} \) and then making \( \epsilon \to 0 \), we get \( \limsup_{r \to \infty} n(r)/r^\lambda(r) \leq M\lambda \). Similarly we can prove that \( \liminf_{r \to \infty} n(r)/r^\lambda(r) \geq M\lambda \), and the first part of the theorem follows. The proof of the second part is similar.

We omit the proofs of Theorems 4 and 5.

5. We know that for all values of \( a \)

\[
(5.1) \quad \limsup_{r \to \infty} \frac{n(r, a)}{r^\rho(r)} < \infty.
\]
The question naturally arises whether (5.1) is still true if we replace \( \rho(r) \) by \( \lambda(r) \). We show that this is not so. Below we give an example in which \( \limsup_{r \to \infty} n(r, a)/r^\lambda(r) = \infty \). Take, \( f(z) = \prod_{k=1}^{\infty} (1 + (z/A^n)^{k\mu}) \) where \( k = [\rho] + 1, \mu_n = \Delta_n^{n+1}, \Delta_n = n^n \). Then,

\[
\limsup_{r \to \infty} n(r, 0)/\log M(r, f) = \infty
\]

(see S. M. Shah [3]). Now, since \( \log M(r, f) \leq r^{\lambda(r)} \) for \( r \geq r_0 \), so

\[
\limsup_{r \to \infty} n(r, 0)/r^{\lambda(r)} = \infty.
\]

References


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