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A RING ADMITTING MODULES OF LIMITED DIMENSION

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Let K be a ring with unit. A module[1] M over K is said to be *finite dimensional* if it (i) is finitely based, and (ii) contains no infinite independent set. For such a module there must exist [1, Theorem 7, p. 245] an integer n such that all bases have length n (the *invariant basis number* property), and no independent set has length greater than n. It was shown in a recent paper [1, Theorem 6, p. 245] that this property carries downward with decreasing length of basis. That is: *If K admits a module of finite dimension n, then every module over K having a basis of length $\leqq n$ is also finite dimensional.*

It was remarked (in [1]) that this leaves open the possibility that a ring could exist admitting only modules of limited dimension. That is, for some fixed integer n there might exist a ring K such that a module over K is finite dimensional if and only if it has a basis of length $\leqq n$. It is the purpose of this paper to construct such a ring for arbitrary n.

Let R be the ring of (noncommutative) polynomials generated over the field of integers modulo 2 by a countably infinite set of symbols $\{x_i, y_j\}$, with $i=1, \cdots, m=(n+2)(n+1); j=1, 2, \cdots$, where n is the fixed integer chosen. Let R' be the subring of R generated by the $\{x_i\}$. It is desired to order a (suitably restricted) set of n-dimensional row vectors of members of R'. Begin by ordering the set of all

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[1] Throughout this paper "module" will mean "left module."

monomials $w = x_{i_1}x_{i_2}\cdots x_{i_t}$, where $t = 1, 2, \cdots$. If $w' = x'_{i_1}x'_{i_2}\cdots x'_{i_s}$, set $w < w'$ if either:

$$\sum i_k < \sum i'_k, \text{ or} \tag{1}$$

$$\sum i_k = \sum i'_k \tag{2}$$

and the sequence $i_1, \cdots, i_t$ precedes $i'_1, \cdots, i'_s$ lexicographically. Also set $1 < w$ for all such w.

LEMMA 1. *If $u \neq 1$, then $w < wu$.*

This is clear from condition (1).

LEMMA 2. *If $uv \leqq wv$, then $u \leqq w$.*

Cancelling the same term from two monomials ordered by (1) and (2) does not change either the comparison in (1) or the lexicographic order in (2).

LEMMA 3. *If $u \leqq u'$ and $v \leqq v'$, then $uv \leqq u'v'$.*

By an argument reversing that of the preceding lemma, multiplication on the right (or left) by the same monomial does not change the ordering. Thus $uv \leqq u'v \leqq u'v'$.

Now let each polynomial $\alpha \in R'$ be written with its terms in descending order. Thus $\alpha = w_1 + \cdots + w_r$, with $w_1 > \cdots > w_r$. Order the polynomials according to lexicographical order of the sequences $w_1, \cdots, w_r$, and set $0 < 1$.

Finally, consider the set of all n-dimensional row vectors $(\gamma_1, \cdots, \gamma_n)$ over R'. A vector is said to be *admissible* if it satisfies the conditions:

(a) At least one $\gamma_i \neq 0$, and all $\gamma_i \neq 1$.

(b) $\gamma_r > \gamma_{r+1}$ if $\gamma_r \neq 0$, and $\gamma_r = \gamma_{r+1}$ if $\gamma_r = 0$, for $r = 1, \cdots, n-1$.

(c) The leading term of a γ_r is not a left divisor of any term of any other γ_s.

LEMMA 4. *Every nonzero vector over R' may be reduced by elementary (column) transformations to either* $(1, 0, \cdots, 0)$ *or an admissible vector.*

Suppose condition (c) is not satisfied. Then there exist $\gamma_r = w_1 + \cdots$ and $\gamma_s = u_1 + \cdots$, with some $u_t = w_1 v$. Then $\gamma_s' = \gamma_s + \gamma_r v = u_1 + \cdots + u_{t-1} + u_{t+1} + \cdots + w_2 v + \cdots$. Since the terms of γ_r and γ_s have been written in descending order, we have $u_t > u_j$ $(j \geqq t+1)$, and by Lemma 3, $u_t = w_1 v > w_i v$ $(i \geqq 2)$. Thus $\gamma_s' < \gamma_s$. If 1 appears at any stage as one of the components, the vector is immediately reducible to $(1, 0, \cdots, 0)$. Otherwise it is clear that the above process must end after a finite number of steps in a vector satisfying (a) and (c). A permutation then yields a vector satisfying all three conditions.

The set of all admissible vectors $(\gamma_1, \cdots, \gamma_n)$ is now ordered according to lexicographic order of the sequences $\gamma_1, \cdots, \gamma_n$. Let $(\gamma_1^k, \cdots, \gamma_n^k)$ with $k=1, 2, \cdots$ designate this ordered set of vectors. Also if $\gamma_j^k \neq 0$, let z_j^k be its leading term.

LEMMA 5. *Any nonzero sum $\sum_{j=1}^n \gamma_j^k \phi_j$, where all $\phi_j \in R'$ and the u_j are the leading terms of the nonzero ϕ_j, has leading term $z_t^k u_t$ for some t.*

If $\gamma_j^k \phi_j \neq 0$, its leading term, by Lemma 3, is $z_j^k u_j$. By condition (c) on admissible vectors, no z_j^k is the left divisor of another, so the $z_j^k u_j$ are all distinct. Thus if $z_t^k u_t$ is maximal among the $z_j^k u_j$, it is greater than all other terms in the sum.

Now let H be the two-sided ideal generated by

$$y_k \gamma_j^k, \tag{3}$$

$$x_i y_h, \tag{4}$$

$$y_k y_h, \tag{5}$$

where $j=1, \cdots, n$; $i=1, \cdots, m$; $k, h=1, 2, \cdots$. The ring whose construction is the purpose of this paper is $K=R/H$, and it will now be shown that K does, indeed, have the desired properties.

We will first show that if a module over K has a basis of length $t \leqq n$, then every set of $t+1$ of its members is dependent. This is equivalent to saying that if A is a $t+1$ by t matrix, then there is a nonzero vector X such that $XA=0$. Remark, first, from (4) and (5), that if a monomial contains a y_h anywhere other than as initial symbol, it is a member of H. If every term of every element of the first row of A begins with some y_h, then we may choose $X=(x_i, 0, \cdots, 0)$ for any x_i. Otherwise, by Lemma 4, an elementary matrix E exists such that the portion of the first row of AE free of y_h either forms the first t elements of some admissible vector $(\gamma_1^k, \cdots, \gamma_t^k, 0, \cdots, 0)$ or is $(1, 0, \cdots, 0)$. In the former case, we choose $X=(y_k, 0, \cdots, 0)$.

In the latter case, let α_{i1} designate the elements of the first column of AE, so that $\alpha_{11}=1+\sum y_h\theta_h$. Now, it is clear that for any $\beta \in K$ we have $(\beta+\phi)\alpha_{11}=\beta$, where $\phi=\sum y_h\theta_h$ if 1 is a term of β, $\phi=0$ otherwise. If we let $\phi_i=\sum y_h\theta_h$ or 0 according as 1 is or is not a term of α_{i1}, then $\alpha_{i1}+(\alpha_{i1}+\phi_i)\alpha_{11}=0$ for $i \geqq 2$. There is accordingly an elementary matrix U such that

$$UAE = \begin{bmatrix} \alpha_{11} & * \\ 0 & A' \end{bmatrix},$$

for some $t-1$ by t matrix A'. Assuming, by way of induction, that there is an X' such that $X'A'=0$, we may take $X=(OX')U$. For the case $t=1$, of course, we have simply

$$UAE = \begin{bmatrix} \alpha_{11} \\ 0 \end{bmatrix},$$

and we can let $X=(0\ 1)U$.

The above discussion establishes the sufficiency of the following:

THEOREM. *For each integer n there exists a ring K such that a necessary and sufficient condition for a module over K to be finite dimensional is that it have a basis of length $\leqq n$.*

To establish the necessity, it is enough to exhibit a module with a basis of length $n+1$ which is not finite dimensional. This will be the space of all $(n+1)$-dimensional vectors, and the independent sets will be the rows of

$$T = \begin{bmatrix} x_1 & x_{n+3} & \cdots & x_{n(n+2)+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{n+2} & x_{2n+4} & \cdots & x_m \end{bmatrix}.$$

Suppose $XT=0$ for some vector X. Clearly $X=X_0+\sum y_kX_k$ for some set of vectors X_0, X_k with elements from R'. From the definition of H, it is clear that $XT=0$ implies $X_0T=0$ and $y_kX_kT=0$ for each k. Again, from the form of T, it follows that $X_0=0$.

Consider a fixed k, and let $X_k=(\phi_1, \cdots, \phi_{n+2})$, $V_k=(\gamma_1^k, \cdots, \gamma_n^k)$. From (3) it is clear that in any relation of form $y_k\psi=0$, the ψ must be a linear combination of the $\gamma_j^k (j=1, \cdots, n)$. Thus from the relation $y_kX_kT=0$ it follows that

$$V_kB = X_kT, \tag{6}$$

for some matrix B with elements from R'. Let $B=[\beta_{st}]$ and set $\beta_{st}=\sum_{i=1}^m \beta_{sti}x_i+\beta_{st0}$, where each $\beta_{st0}=0$ or 1. Similarly set $\gamma_j^k=\sum_{i=1}^m \gamma_{ji}x_i+\gamma_{j0}$.

Now, from the form of T, we have $X_kT=(\alpha_0, \cdots, \alpha_n)$, where $\alpha_h=\sum_{i=1}^{n+2} \phi_i x_{h(n+2)+i}$ for $h=0, 1, \cdots, n$. Equating coefficients of x_r with $r=1, \cdots, m$ in (6), we have for each h

$$L_{hr} = \begin{cases} \phi_i & \text{if } r = h(n+2)+i \text{ with } 1 \leqq i \leqq n+2, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where

$$L_{hr} = \sum_{j=1}^{n} \gamma_j^k \beta_{jhr} + \sum_{j=1}^{n} \gamma_{jr}\beta_{jh0}.$$

Let x_{r_j} be the last symbol of z_j^k and set $z_j^k = v_j x_{r_j}$. Now for each r_j there is exactly one value of h satisfying $r_j = h(n+2)+i$ for some i in the range $1 \leqq i \leqq n+2$. Since there are at most n distinct values of r_j, whereas h has $n+1$ values, there must be at least one value of h such that for all r_j we have $r_j \neq h(n+2)+i$. For this value of h we have $L_{hr_j} = 0$ for all r_j.

Now assume that there is some $s \geqq 1$ for which $\beta_{sh0} \neq 0$, while $\beta_{jh0} = 0$ for all $j < s$. The sum $\sum_{j=1}^{n} \gamma_{jr_s}\beta_{jh0}$ collapses to $\sum_{j=s}^{n} \gamma_{jr_s}\beta_{jh0}$. Since z_j^k is a term (in fact, the leading term) of γ_j^k, and since it has final symbol x_{r_j}, it is contained in $\gamma_{jr_j}x_{r_j}$. Further $\gamma_s^k > \gamma_j^k$ (when $j > s$) and so its leading term, z_s^k, is greater than all other terms of γ_j^k $(j \geqq s)$. In particular, z_s^k is greater than all terms of $\gamma_{jr_s}x_{r_s}$ $(s > j)$. By Lemma 2, v_s is greater than all terms of γ_{jr_s} and so is the leading term of $\sum_{j=s}^{n} \gamma_{jr_s}\beta_{jh0}$. Now by Lemma 5, the leading term of $\sum_{j=1}^{n} \gamma_j^k \beta_{jhr_s}$ is $z_t^k u_t$ for some integer t. Since $L_{hr_s} = 0$ it follows that $z_t^k u_t + v_s = 0$. But then z_t^k would be a left divisor of z_s^k, violating condition (c) on admissibility. From this contradiction, we must conclude that $\beta_{jh0} = 0$ for all $j = 1, \cdots, n$. From (7) and (8) it then follows that each ϕ_i, and hence X_k, is a sum of multiples of the γ_j^k. Thus $y_k X_k = 0$, and since k was arbitrary, we conclude that $X = 0$.

One additional remark may be made concerning the ring K. While K admits finitely based modules which are not finite dimensional, it is nevertheless true that all such modules satisfy the invariant basis number property. This is equivalent to saying that all unimodular matrices over K are square, which is a consequence of the fact that none of the generators, (1), (2), or (3) of H contain the term 1. (See the argument in [**2**, preceding Theorem 1, p. 189].) It is an open question whether or not there exists a ring satisfying both the above theorem and the condition that its modules above some level lack the invariant basis number property.

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