# INVERSION OF AN INTEGRAL TRANSFORM[1]

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1. **Introduction.** It is the object of this note to give an inversion formula for the integral transform,

$$f(x) = \int_0^\infty k(y)\phi(x+y)dy, \qquad x > 0 \tag{1}$$

under the assumptions:

(i) $k \in L(0, \infty)$;

(ii) $k \not\equiv 0$ in the neighborhood of zero;

(iii) the Laplace transform

$$K(s) = \int_0^\infty e^{-sx}k(x)dx$$

has no zeros in the closed right half-plane;

(iv) $\phi \in L_p(0, \infty)$ for some $p$ in $1 \leqq p \leqq 2$.

Formally such a formula can be obtained as follows. Define $\phi$ and $k$ to be zero for negative values of the argument, and define $f$ by the equation in (1) for all real $x$. If we denote Fourier transformation by a circumflex, i.e.

$$\hat{g}(t) = \int_{-\infty}^\infty e^{itx}g(x)dx$$

it follows from (1) that

$$\hat{f}(t) = \hat{k}(-t)\hat{\phi}(t).$$

Hence

$$\begin{aligned}\phi(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-itx}\hat{\phi}(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-itx}\,\frac{\hat{f}(t)}{\hat{k}(-t)}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-itx}\frac{dt}{\hat{k}(-t)}\int_{-\infty}^\infty e^{ity}\,f(y)dy \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f(x+y)dy\int_{-\infty}^\infty \frac{e^{ity}}{\hat{k}(-t)}\,dt.\end{aligned}$$

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This last formula is clearly unsatisfactory since $f(x)$ is given only for $x>0$. However under our assumptions on $k$ it happens that

$$\int_{-\infty}^{\infty} \frac{e^{ity}}{\hat{k}(-t)}\,dt = 0, \qquad y < 0,$$

so the formula becomes

$$\phi(x) = \frac{1}{2\pi}\int_0^{\infty} f(x+y)dy \int_{-\infty}^{\infty} \frac{e^{ity}}{\hat{k}(-t)}\,dt.$$

The actual results which we shall prove, motivated by this formula, are these

THEOREM 1. *Under hypotheses* (i)–(iv) *the equation* (1) *is inverted by*

$$\phi(x) = \underset{\delta\to 0+}{\overset{(p)}{\text{l.i.m.}}} \lim_{\epsilon\to 0+} \frac{1}{2\pi}\int_0^{\infty} f(x+y)dy \int_{-\infty}^{\infty} \frac{e^{it(y-\delta)-\epsilon|t|}}{\hat{k}(-t)}\,dt.$$

THEOREM 2. *Under hypotheses* (i)–(iv) *the equation* (1) *is inverted by*

$$\phi(x) = \lim_{\delta\to 0+} \lim_{\epsilon\to 0+} \frac{1}{2\pi}\int_0^{\infty} f(x+y)dy \int_{-\infty}^{\infty} \frac{1-e^{-i\delta t}}{i\delta t}\, \frac{e^{ity-\epsilon|t|}}{\hat{k}(-t)}\,dt,$$

*for almost all* $x$, *and at all points of right-continuity of* $\phi$.

An alternative technique, suggested by Sparenberg [**3**], seems difficult to apply.

2. **Lemmas.** We need Hayman's extension [**1**, Theorem 2] of the Ahlfors-Heins principle.[2]

LEMMA 1. *If* $u(z)$ *is subharmonic in the half-plane* $y>0$ *and if*

(a) $$\limsup_{z\to x} u(z) \leqq 0,$$

$$\alpha = \sup_{y>0} \frac{u(x+iy)}{y}$$

(b) $$\alpha_0 = \limsup_{r\to\infty} \frac{1}{r} \sup_{|z|=r;y>0} u(z) < \infty$$

*then* $\alpha_0 = \max(\alpha, 0)$ *and* $\lim u(re^{i\theta})/r = \alpha \sin\theta$ *uniformly for* $0<\theta<\pi$ as $r\to\infty$ *omitting an r-set of finite logarithmic length* (*i.e., a set E with* $\int_E r^{-1}dr < \infty$).

[2] The authors are indebted to Professor W. H. J. Fuchs for calling their attention to this lemma, basic for their results.

LEMMA 2. *Under the hypotheses* (i)–(iii) *imposed on k the function* $1/\hat{k}(z)$ *has the property*

$$\frac{1}{\hat{k}(z)} = O(e^{\epsilon|z|}), \qquad \epsilon > 0$$

*on a sequence of semi-circular arcs* $|z| = r_n$, $0 \leqq \theta \leqq \pi$, *with* $r_n \to \infty$ *as* $n \to \infty$.

Note that $\hat{k}(z) = K(-iz)$ so that $1/\hat{k}(z) \neq 0$ for $y \geqq 0$. We may suppose that $|\hat{k}(z)| \leqq 1$ for $y \geqq 0$.

It follows from hypotheses (i)–(iii) that $\hat{k}$ has the representation

$$u(z) = \log |\hat{k}(x + iy)| = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log |\hat{k}(v)|}{y^2 + (x - v)^2} dv$$

for $y > 0$. See, for example, Nyman [2, pp. 13, 29].

Since $\log |\hat{k}(v)| \leqq 0$ it follows from the representation that $u(z) \leqq 0$ and also that $\lim_{y\to\infty} u(z)/y = 0$. This confirms hypothesis (a) of Lemma 1 and shows that $\alpha = 0$. (b) is also a consequence of the representation.

Hence

$$\lim_{r\to\infty} \frac{u(re^{i\theta})}{r} = 0$$

with the prescribed uniformity. This establishes the lemma.

LEMMA 3. *Under the hypotheses* (i)–(iii) $\lim_{u\to\infty} \log |\hat{k}(iu)|/u = 0$.

This is an immediate consequence of the representation of $u(z)$.

3. **The inversion formulas.** It follows from (1) and the hypotheses (i), (iv) that

$$\hat{f}(t) = \hat{k}(-t)\hat{\phi}(t)$$

where the Fourier transforms are taken in the appropriate sense. Therefore (by imitating the proof of Theorem 59 in Titchmarsh [4]) we have

$$\phi(x) = \lim_{\epsilon\to 0+} \phi_\epsilon(x) \qquad \text{p.p.}$$

where

$$\phi_\epsilon(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-\epsilon|t|} \frac{\hat{f}(t)}{\hat{k}(-t)} dt.$$

Hence

$$\phi_\epsilon(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}e^{-\epsilon|t|}}{\hat{k}(-t)}\, dt \ \underset{a\to\infty}{\overset{(q)}{\text{l.i.m.}}} \int_{-a}^{a} e^{ity}f(y)dy$$

$$(2) \qquad = \lim_{a\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}e^{-\epsilon|t|}}{\hat{k}(-t)}\, dt \int_{-a}^{a} e^{ity} f(y)dy$$

$$(3) \qquad = \lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a} f(y)dy \int_{-\infty}^{\infty} \frac{e^{it(y-x)}e^{-\epsilon|t|}}{\hat{k}(-t)}\, dt$$

$$(4) \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+y)dy \int_{-\infty}^{\infty} \frac{e^{ity}e^{-\epsilon|t|}}{\hat{k}(-t)}\, dt,$$

where $q = p/(p-1)$. (2) is justified since $e^{-\epsilon|t|}/\hat{k}(-t)$ is in $L_p(-\infty, \infty)$, (3) by Fubini's theorem, and (4) because the inner integral as a function of $y$ is in $L_q(-\infty, \infty)$.

Now let

$$R_\epsilon(y) = \int_{-\infty}^{\infty} \frac{e^{ity-\epsilon|t|}}{\hat{k}(-t)}\, dt$$

and consider the integral of

$$\frac{e^{iwy-\epsilon w}}{\hat{k}(-w)},$$

as a function of $w$, over the contour consisting of the intervals $(0, R)$ and $(0, -iR)$ and the quadrant of the circle joining $R$ and $-iR$. If $y<0$ it follows by Lemma 2 that the integral over the quadrant approaches zero as $R = r_n \to \infty$. Therefore

$$\int_0^{\infty} \frac{e^{ity-\epsilon t}}{\hat{k}(-t)}\, dt = \lim_{n\to\infty}\int_0^{r_n} \frac{e^{uy+i\epsilon u}}{\hat{k}(iu)}\, du.$$

Similarly

$$\int_{-\infty}^{0} \frac{e^{ity+\epsilon t}}{\hat{k}(-t)}\, dt = -\lim_{n\to\infty}\int_0^{r_n} \frac{e^{uy-i\epsilon u}}{\hat{k}(iu)}\, du.$$

Thus for $y<0$

$$R_\epsilon(y) = 2i\lim_{n\to\infty}\int_0^{r_n} \frac{e^{uy}\sin \epsilon u}{\hat{k}(iu)}\, du = 2i\int_0^{\infty} \frac{e^{uy}\sin \epsilon u}{\hat{k}(iu)}\, du.$$

The last step is justified by Lemma 3 which enables us to conclude also that

$$\underset{\epsilon\to 0+}{\overset{(q)}{\text{l.i.m.}}}\ R_\epsilon(y) = 0 \qquad \text{on } (-\infty, -\delta)$$

for each $\delta > 0$. According to (4)

$$\phi(x) = \lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x+y) R_\epsilon(y) dy$$

and so

$$\phi(x) = \lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_{-\delta}^{\infty} f(x+y) R_\epsilon(y) dy \qquad \text{p.p.}$$

for each $\delta > 0$. Consequently for each $\delta > 0$

$$\phi(x+\delta) = \lim_{\epsilon \to 0} + \frac{1}{2\pi} \int_{0}^{\infty} f(x+y) R_\epsilon(y-\delta) dy \tag{5}$$

and Theorem 1 follows.

We turn to the proof of Theorem 2. We have (by imitating the proof of Theorem 19 of Titchmarsh [**4**]) $\phi(x) = \text{l.i.m.}^{(p)}_{\epsilon \to 0+} \phi_\epsilon(x)$, so for any fixed $x$, $\phi(x+h) = \text{l.i.m.}^{(p)}_{\epsilon \to 0+} \phi_\epsilon(x+h)$ over any $h$-interval $(0, \delta)$. Therefore, by the argument leading to (5),

$$\phi(x+h) = \underset{\epsilon \to 0+}{\overset{(1)}{\text{l.i.m.}}} \frac{1}{2\pi} \int_{0}^{\infty} f(x+y) R_\epsilon(y-h) dy,$$

so

$$\begin{aligned} \int_0^\delta \phi(x+h) dh &= \lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_0^\delta dh \int_0^\infty f(x+y) R_\epsilon(y-h) dy \\ &= \lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_0^\infty f(x+y) dy \int_0^\delta R_\epsilon(y-h) dh \\ &= \lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_0^\infty f(x+y) dy \int_{-\infty}^{\infty} \frac{1-e^{i\delta t}}{i\delta t} \frac{e^{ity-\epsilon|t|}}{\hat{k}(-t)} dt \end{aligned}$$

and Theorem 2 follows.

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