A NOTE ON THE RELATIONSHIP BETWEEN CERTAIN SUBGROUPS OF A FINITE GROUP

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A well-known result of G. Frobenius (cf. [2]) states that if $\mathfrak{H}$ is a normal subgroup of the finite group $\mathfrak{G}$, then an irreducible $\mathfrak{G}$-module (relative to any base field $\mathfrak{F}$) either remains irreducible as an $\mathfrak{H}$-module or decomposes into a direct sum of conjugate irreducible $\mathfrak{H}$-modules. Simple examples readily demonstrate that the conclusion of this theorem may hold even though $\mathfrak{H}$ is not normal. In §1 a version of the Frobenius result is stated and the converse considered. This opens the question: What is the relationship between a group $\mathfrak{G}$ and one of its subgroups $\mathfrak{H}$ if each irreducible $\mathfrak{G}$-module over a field $\mathfrak{F}$ remains irreducible as an $\mathfrak{H}$-module? It is shown in §2 that for "most" fields $\mathfrak{F}$ (the modular fields naturally cause a certain amount of difficulty) the answer is that $\mathfrak{G}$ is an extension of $\mathfrak{H}$ by an abelian group such that each conjugate class of $\mathfrak{H}$ is also a conjugate class of $\mathfrak{G}$. To determine whether this last property leads to the conclusion that $\mathfrak{G}$ is the trivial extension of $\mathfrak{H}$, extensions are considered in §3 and it is shown that the answer is in general negative. However, using a result due to M. Hall [4] it is proved that this latter property does imply that $\mathfrak{G}$ is the trivial extension of $\mathfrak{H}$ in many cases.

Since results contingent on absolute irreducibility are used in certain proofs,[1] it will be assumed throughout this note that $\mathfrak{F}$ is always a splitting field for every irreducible representation of the groups being discussed.

1. **Preliminary remarks.** Let $\mathfrak{H}$ be a subgroup of the finite group $\mathfrak{G}$ and let $\mathfrak{M}$ be a left (right) $\mathfrak{G}$-module with base field $\mathfrak{F}$. If $\mathfrak{N}$ is a left (right) $\mathfrak{H}$-submodule of $\mathfrak{M}$ and if $G\in\mathfrak{G}$, then submodule $G\mathfrak{N}(\mathfrak{N}G)$ of $\mathfrak{M}$ is said to be a conjugate of $\mathfrak{N}$ relative to $\mathfrak{G}$. Obviously it need not be an $\mathfrak{H}$-module.

Now the key to the Frobenius Theorem is the result [2]:

If $\mathfrak{H}$ is a normal subgroup of $\mathfrak{G}$ then an irreducible $\mathfrak{G}$-module $\mathfrak{M}$ contains an irreducible $\mathfrak{H}$-submodule $\mathfrak{N}$ which has the property that each conjugate of $\mathfrak{N}$ relative to $\mathfrak{G}$ is also an $\mathfrak{H}$-module.

Consideration of the converse proposition leads to the following:

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[1] It was pointed out by the referee that Theorem 3, for example, may be false if $\mathfrak{F}$ is not a splitting field for every irreducible representation of G and H. The symmetric group on three elements, its normal subgroup, and the rational field illustrate this possibility.

THEOREM 1. *If $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$ such that each irreducible $\mathfrak{G}$-module $\mathfrak{M}$ over a field $\mathfrak{F}$ contains an irreducible $\mathfrak{H}$-submodule $\mathfrak{U}$ all of whose conjugates relative to $\mathfrak{G}$ are also $\mathfrak{H}$-modules, then each irreducible $\mathfrak{K}$-module remains irreducible as an $\mathfrak{H}$-module, where $\mathfrak{K}$ is the minimal normal subgroup of $\mathfrak{G}$ which contains $\mathfrak{H}$.*

Let $\mathfrak{M}$ be an irreducible left $\mathfrak{G}$-module. Since $\mathfrak{K}$ is normal in $\mathfrak{G}$, $\mathfrak{M}$ is a direct sum of conjugate irreducible left $\mathfrak{K}$-modules, $\mathfrak{N}_i$, each of dimension m relative to $\mathfrak{F}$: $\mathfrak{M}=\mathfrak{N}_1+\cdots+\mathfrak{N}_n$, $n\geqq 1$. On the other hand, from the hypothesis $\mathfrak{M}$ contains an irreducible left $\mathfrak{H}$-submodule $\mathfrak{U}$ all of whose conjugates relative to $\mathfrak{G}$ are also $\mathfrak{H}$-modules, necessarily irreducible. Now let $G\in\mathfrak{G}$, $H\in\mathfrak{H}$; then $G\mathfrak{U}$ is an $\mathfrak{H}$-module and therefore $(G^{-1}HG)\mathfrak{U}=G^{-1}H(G\mathfrak{U})=G^{-1}(G\mathfrak{U})=\mathfrak{U}$. So $\mathfrak{U}$, of dimension u over $\mathfrak{F}$, is an irreducible $\mathfrak{K}$-module. Therefore $m=u$ and since each $\mathfrak{N}_i$ is also a left $\mathfrak{H}$-module it must remain irreducible as an $\mathfrak{H}$-module. As every irreducible $\mathfrak{K}$-module is $\mathfrak{K}$-isomorphic with a submodule of a $\mathfrak{G}$-module, the result follows.

This interesting relationship between $\mathfrak{H}$ and $\mathfrak{K}$ will be investigated in the remainder of the paper.

2. **Property $\mathcal{I}$.** To simplify matters we introduce the following definition. A subgroup $\mathfrak{H}$ of the group $\mathfrak{G}$ is said to possess property $\mathcal{I}$ relative to the field F if each irreducible $\mathfrak{G}$-module over $\mathfrak{F}$ remains irreducible as an $\mathfrak{H}$-module.

THEOREM 2. *If $\mathfrak{H}$ possesses property $\mathcal{I}$ relative to $\mathfrak{F}$ then $\mathfrak{H}$ is normal in $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{H}$ is an abelian group if either of the following conditions is satisfied*:

(i) *The radical $\mathfrak{R}(\mathfrak{G})$ of the group algebra $\mathfrak{A}(\mathfrak{G})$ of $\mathfrak{G}$ over F equals $\mathfrak{A}(\mathfrak{G})\cdot\mathfrak{R}(\mathfrak{H})$, where $\mathfrak{R}(\mathfrak{H})$ is the radical of $\mathfrak{A}(\mathfrak{H})$, the group algebra of $\mathfrak{H}$ over $\mathfrak{F}$.*

(ii) *The characteristic of $\mathfrak{F}$ is p and $\mathfrak{H}$ is a Sylow p-subgroup of $\mathfrak{G}$.*

Let $\mathfrak{H}$ be the ideal of $\mathfrak{A}(\mathfrak{H})$ which has as its basis the differences H_i-H_j, all H_i, $H_j\in\mathfrak{H}$. Then $\mathfrak{H}$ is normal in $\mathfrak{G}$ if and only if the left ideal $\mathfrak{L}=\mathfrak{A}(\mathfrak{G})\cdot\mathfrak{H}$ is a two-sided ideal in $\mathfrak{A}(\mathfrak{G})$. Now (i) implies that $\mathfrak{L}\supseteq\mathfrak{R}(\mathfrak{G})$ since $\mathfrak{H}\supseteq\mathfrak{R}(\mathfrak{H})$, so it will be sufficient to show that the image $\overline{\mathfrak{L}}$ of $\mathfrak{L}$ in $\overline{\mathfrak{A}}(\mathfrak{G})=\mathfrak{A}(\mathfrak{G})-\mathfrak{R}(\mathfrak{G})$ is an ideal. $\overline{\mathfrak{A}}(\mathfrak{G})$ contains an algebra $\mathfrak{D}\cong\overline{\mathfrak{A}}(\mathfrak{H})$ and $\mathfrak{D}=\mathfrak{U}\oplus\mathfrak{B}$ with $\mathfrak{U}\cong\mathfrak{A}(\mathfrak{H})-\mathfrak{H}$ and of dimension one over $\mathfrak{F}$. Then $\overline{\mathfrak{A}}(\mathfrak{G})=\overline{\mathfrak{A}}(\mathfrak{G})\mathfrak{D}=\overline{\mathfrak{A}}(\mathfrak{G})\mathfrak{U}+\overline{\mathfrak{A}}(\mathfrak{G})\mathfrak{B}$, a direct sum of left ideals of $\overline{\mathfrak{A}}(\mathfrak{G})$, with $\overline{\mathfrak{A}}(\mathfrak{G})\mathfrak{B}\cong\overline{\mathfrak{L}}$. But $\overline{\mathfrak{A}}(\mathfrak{G})\mathfrak{U}$ and $\overline{\mathfrak{L}}$ are right $\mathfrak{H}$-modules, so if $\mathfrak{B}$ is a minimal right ideal of $\overline{\mathfrak{A}}(\mathfrak{G})$, hence an irreducible right $\mathfrak{H}$-module, it must lie entirely in $\overline{\mathfrak{A}}(\mathfrak{G})\mathfrak{U}$ or $\overline{\mathfrak{L}}$. Hence $\overline{\mathfrak{L}}$

is also a right ideal of $\bar{\mathfrak{A}}(\mathcal{G})$ and so $\mathcal{H}$ is normal in $\mathcal{G}$. Furthermore $\mathcal{G}/\mathcal{H}$ is represented isomorphically over $\mathfrak{A}(\mathcal{G})-\mathfrak{L}\cong\bar{\mathfrak{A}}(\mathcal{G})\mathfrak{U}$ which is necessarily a sum of fields since $\mathfrak{U}$ is one dimensional.

If (ii) is satisfied then all the irreducible representations of G are one dimensional since the only irreducible representation of $\mathcal{H}$ is the identity representation. Therefore there exists a minimal normal subgroup $\mathcal{K}$ such that $\mathcal{G}/\mathcal{K}$ is abelian and $\mathfrak{A}(\mathcal{G}/\mathcal{K})$ is semisimple. It follows simply (cf. [**3**]) that $\mathcal{K}$ is necessarily of order p^a and hence $\mathcal{K}=\mathcal{H}$.

If $\mathfrak{F}$ is restricted so that $\mathfrak{A}(\mathcal{G})$ is semisimple then the following deeper result may be obtained.

THEOREM 3. *If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ possessing property $\mathcal{I}$ relative to the field $\mathfrak{F}$ of characteristic* 0 *or p, $(p, o(G))=1$, then each conjugate class of $\mathcal{H}$ is also a conjugate class in $\mathcal{G}$.*

Let $\mathfrak{C}(\mathcal{G})$ and $\mathfrak{C}(\mathcal{H})$ be the centers of $\mathfrak{A}(\mathcal{G})$ and $\mathfrak{A}(\mathcal{H})$ respectively. We must show that $\mathfrak{C}(\mathcal{H})$ is a subalgebra of $\mathfrak{C}(\mathcal{G})$. Let $\mathfrak{L}$ be a minimal left ideal of $\mathfrak{A}(\mathcal{G})$; hence it is an irreducible left $\mathcal{H}$-module and so there exists a primitive idempotent $e\in\mathfrak{C}(\mathcal{H})$ such that $e\mathfrak{A}(\mathcal{H})\mathfrak{L}=\mathfrak{L}$. Now $\mathfrak{A}(\mathcal{G})=\mathfrak{A}(\mathcal{H})\cdot\mathfrak{A}(\mathcal{G})$ and $e\mathfrak{A}(\mathcal{H})\mathfrak{A}(\mathcal{G})\supseteq\mathfrak{L}$, so if $\mathcal{L}$ is the set of all minimal left ideals $\mathfrak{L}$ of $\mathfrak{A}(\mathcal{G})$ such that $e\mathfrak{A}(\mathcal{H})\mathfrak{L}=\mathfrak{L}$, then $e\mathfrak{A}(\mathcal{H})\mathfrak{A}(\mathcal{G})=\bigcup_{\mathfrak{L}\in\mathcal{L}}\mathfrak{L}$. Since $\mathfrak{C}(\mathcal{H})$ may be written as $(e_1)\oplus\cdots\oplus(e_m)$, each e_i a primitive idempotent, then $\mathfrak{A}(\mathcal{G})=e_1\mathfrak{A}(\mathcal{H})\mathfrak{A}(\mathcal{G})+\cdots+e_m\mathfrak{A}(\mathcal{H})\mathfrak{A}(\mathcal{G})$ is a direct decomposition of $\mathfrak{A}(\mathcal{G})$ into left ideals. Observing that $e_i-Ge_iG^{-1}$ annihilates $\mathfrak{A}(\mathcal{G})$ from the left, any $G\in\mathcal{G}$, we conclude that $\mathfrak{C}(\mathcal{H})\subset\mathfrak{C}(\mathcal{G})$.

Indicative of the inconclusiveness of the modular case is

THEOREM 4. *If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ possessing property $\mathcal{I}$ over the field $\mathfrak{F}$ of characteristic p and if all the irreducible representations of $\mathcal{H}$ over $\mathfrak{F}$ are one dimensional, then $\mathcal{G}$ is an extension of a p-group by an abelian group of order q, $(q, p)=1$. Conversely, if $\mathcal{G}$ is an extension of a p-group by an abelian group of order q, $(q, p)=1$, then any subgroup $\mathcal{H}$ of $\mathcal{G}$ possesses property $\mathcal{I}$ relative to a field of characteristic p.*

Since an irreducible $\mathcal{H}$-module has dimension one, property $\mathcal{I}$ implies that each irreducible $\mathcal{G}$-module is one dimensional over $\mathfrak{F}$. Therefore $\bar{\mathfrak{A}}(\mathcal{G})=\mathfrak{A}(\mathcal{G})-\mathfrak{R}(\mathcal{G})$ is a commutative algebra. If $\mathcal{G}'$ is the commutator subgroup of $\mathcal{G}$ and if $\mathfrak{T}$ is the ideal of $\mathfrak{A}(\mathcal{G})$ generated by the differences G_i-G_j, all G_i, $G_j\in\mathcal{G}'$, then clearly $\mathfrak{T}\subseteq\mathfrak{R}(\mathcal{G})$. This means that $\mathcal{G}'$ is a p-group (cf. [**3**]), and the remainder of the theorem is obvious.

Throughout the remainder of the paper the field $\mathfrak{F}$ will be assumed to have characteristic 0 or p with $(p, g)=1$, g the order of $\mathcal{G}$. Then the next result completely characterizes property $\mathcal{J}$ over $\mathfrak{F}$.

THEOREM 5. *Let $\mathcal{H}$ be a normal subgroup of $\mathcal{G}$ of order h and let $\mathcal{H}$ contain s $\mathcal{H}$-conjugate classes. Then $\mathcal{H}$ possesses property $\mathcal{J}$ over $\mathfrak{F}$ if and only if $\mathcal{G}$ contains ns $\mathcal{G}$-conjugate classes, where $g=hn$.*

Let e be a primitive idempotent from the center of $\mathfrak{A}(\mathcal{H})$. Then $\mathfrak{T}=e\mathfrak{A}(\mathcal{H})$ is a minimal two-sided ideal of $\mathfrak{A}(\mathcal{H})$ of order t^2. If $\mathcal{H}$ possesses property $\mathcal{J}$, then by Theorem 3 e is a central idempotent of $\mathfrak{A}(\mathcal{G})$ and therefore $\mathfrak{B}=e\mathfrak{A}(\mathcal{H})\mathfrak{A}(\mathcal{G})$ is a two-sided ideal of $\mathfrak{A}(\mathcal{G})$ of order nt^2. Since $\mathfrak{T}$ is orthogonal with $\mathfrak{A}(\mathcal{H})-\mathfrak{T}$ it follows that each minimal $\mathcal{H}$-submodule of $\mathfrak{B}$ is isomorphic with a minimal $\mathcal{H}$-submodule of $\mathfrak{T}$ and hence is of order t. Then it follows from property $\mathcal{J}$ that each minimal left or right ideal of $\mathfrak{B}$ is of order t, and therefore $\mathfrak{B}$ is expressible as a direct sum of n two-sided ideals of $\mathfrak{A}(\mathcal{G})$, each of order t^2. Since the dimension of the center of $\mathfrak{A}(\mathcal{H})$ is s this implies that $\mathfrak{A}(\mathcal{G})$ decomposes into a direct sum of ns minimal ideals. Hence $\mathcal{G}$ contains ns conjugate classes.

Conversely, suppose $\mathcal{G}$ possesses ns conjugate classes. Since $\mathcal{H}$ has s conjugate classes, $\mathfrak{A}(\mathcal{H})=\mathfrak{T}_1\oplus \cdots \oplus\mathfrak{T}_s$ and this decomposition is unique. Now if $G\in\mathcal{G}$, $A\in\mathfrak{A}(\mathcal{H})$, the mapping $\theta_G\colon A\to A^G=GAG^{-1}$ is an automorphism of $\mathfrak{A}(\mathcal{H})$ and $\mathfrak{T}_i^G$ is a minimal ideal $\mathfrak{T}_j$ of $\mathfrak{A}(\mathcal{H})$. Therefore, under the set of all automorphisms induced by inner automorphisms of $\mathcal{G}$, the minimal ideals $\mathfrak{T}$ of $\mathfrak{A}(\mathcal{H})$ separate into non-overlapping sets of transitivity, $S_1, \cdots, S_m$. That is, if S_i consists of the ideals $\mathfrak{T}_{i,1}, \cdots, \mathfrak{T}_{i,d(i)}$, then $\mathfrak{T}_{ij}^G=\mathfrak{T}_{ik}$, $1\leqq k\leqq d(i)$, for any $G\in\mathcal{G}$, and given any pair $\mathfrak{T}_{ip}$ and $\mathfrak{T}_{iq}$ there exists an element G in $\mathcal{G}$ such that $\mathfrak{T}_{iq}=\mathfrak{T}_{ip}^G$. Then $\mathfrak{B}_i=(\mathfrak{T}_{i,1}+\cdots+\mathfrak{T}_{i,d(i)})\mathfrak{A}(\mathcal{G})$ is a two-sided ideal of $\mathfrak{A}(\mathcal{G})$ of order $nt_i^2d(i)$, t_i^2 the order of $\mathfrak{T}_{ij}$.

Let $\mathfrak{L}$ be a minimal left ideal of $\mathfrak{B}_i$. Then $\mathfrak{T}_{ij}\mathfrak{L}\neq(0)$ for some j and therefore, because of the transitivity of S_i, for all j. Since $\mathfrak{T}_{ij}\mathfrak{L}$ is necessarily of order $\geqq t_i$ and since $\mathfrak{T}_{ij}\mathfrak{T}_{ip}=\delta_{jq}\mathfrak{T}_{ij}$, this implies that the order of $\mathfrak{L}$ is $\geqq t_id(i)$. Therefore a minimal two-sided ideal of $\mathfrak{B}_i$ is of order $\geqq t_i^2[d(i)]^2$, and so no decomposition of $\mathfrak{B}_i$ contains more than $n/d(i)$ two-sided ideals. Therefore $\mathfrak{A}(\mathcal{G})$ decomposes into a sum of not more than $n(1/d(1)+\cdots+1/d(m))$ minimal ideals. However, since $\mathcal{G}$ contains ns conjugate classes, $\mathfrak{A}(\mathcal{G})$ decomposes into a direct sum of ns minimal ideals. Hence $d(1)=\cdots=d(m)=1$, $m=s$, and each minimal left ideal $\mathfrak{L}$ of $\mathfrak{B}_i$ is of order t_i. Since $\mathfrak{L}$ is a left $\mathfrak{T}_i$-module whose order equals the order of a minimal left ideal of $\mathfrak{T}_i$ it follows that $\mathcal{H}$ possesses property $\mathcal{J}$.

Berman has proved [1] that if $\mathfrak{H}$ is a normal subgroup of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{H}$ is cyclic of order n and if each of $s\mathfrak{G}$-conjugate classes C_i contained in $\mathfrak{H}$ splits into h_i $\mathfrak{H}$-conjugate classes, then $\mathfrak{G}$ contains $n(h_1^{-1}+\cdots+h_s^{-1})$ conjugate classes. This result and the previous theorem yield a partial converse to Theorem 3:

THEOREM 6. *If $\mathfrak{G}$ is an extension of $\mathfrak{H}$ by a cyclic group and if each conjugate class of $\mathfrak{H}$ is also a conjugate class of $\mathfrak{G}$, then $\mathfrak{H}$ possesses property $\mathfrak{s}$ over $\mathfrak{F}$.*

3. **Group extensions by abelian groups.** Obviously the trivial extension $\mathfrak{G}$ of a group $\mathfrak{H}$ by an abelian group $\mathfrak{Q}$, $\mathfrak{G}=\mathfrak{H}\times\mathfrak{Q}$, contains a normal subgroup $\mathfrak{H}'\cong\mathfrak{H}$ possessing property $\mathfrak{s}$ over F. Is the trivial extension the only one for which this is so? We shall see that the answer to this depends on whether or not the order c of $\mathfrak{H}$ is prime to the order n of $\mathfrak{G}/\mathfrak{H}$.

If $\mathfrak{H}$ possesses property $\mathfrak{s}$ in $\mathfrak{G}$ then we have seen that $\mathfrak{H}$ is normal in $\mathfrak{G}$ and that $\mathfrak{G}$ induces class-preserving automorphisms on $\mathfrak{H}$. Then the additional condition, $(c, n)=1$, permits us to apply a result due to M. Hall [4, Theorem 6.1] and to conclude that $\mathfrak{G}$ is a trivial extension of $\mathfrak{H}$.

In the other direction we prove the following:

LEMMA. *If $\mathfrak{H}$ is a group containing a q-subgroup $\mathfrak{A}$, q a prime, in its center, then there exists a nontrivial extension $\mathfrak{G}$ of $\mathfrak{H}$ such that $\mathfrak{G}$ contains a subgroup $\mathfrak{H}'\cong\mathfrak{H}$ possessing property $\mathfrak{s}$ in $\mathfrak{G}$, $\mathfrak{G}/\mathfrak{H}'$ of order q.*

Let A be a generator of a cyclic q-subgroup of $\mathfrak{H}$ which is of maximal order q^r among those contained in the center of $\mathfrak{H}$. Let x be an indeterminate and define $\mathfrak{G}$ to be the set of all ordered pairs (x^i, H) where $0\leqq i<q$, $x^0=1$, and H is an element of $\mathfrak{H}$. Then multiplication in $\mathfrak{G}$ is determined by the following definitions: $(x, H_0)^q=(1, A)$, where H_0 is the identity element of $\mathfrak{H}$, and $(x^i, H_j)(x^k, H_n)=(x^m, A^tH_jH_n)$ where $i+j=m+tq$, $0\leqq m<q$. It is easy to verify that $\mathfrak{G}$ is a group containing a subgroup $\mathfrak{H}'=(1, \mathfrak{H})\cong\mathfrak{H}$ possessing property $\mathfrak{s}$ in $\mathfrak{G}$. Furthermore $\mathfrak{G}$ is not isomorphic with the trivial extension of $\mathfrak{H}$ since it contains a cyclic q-subgroup of order q^{r+1} in its center.

To summarize these results:

THEOREM 7. *If a subgroup $\mathfrak{H}$ of a group $\mathfrak{G}$ possesses property $\mathfrak{s}$ relative to $\mathfrak{F}$ then $\mathfrak{G}$ may be a nontrivial extension of $\mathfrak{H}$ but only if the order of $\mathfrak{G}/\mathfrak{H}$ is not prime to the order of $\mathfrak{H}$.*

References

1. S. D. Berman, *Group algebras of Abelian extensions of finite groups*, Dokl. Akad. Nauk. SSSR (N.S.) vol. 102 (1955) pp. 431–434; Math. Rev. vol. 17 (1956) p. 235.

2. A. H. Clifford, *Representations induced in invariant subgroups*, Ann. of Math. vol. 38 (1937) pp. 533–548.

3. W. E. Deskins, *Finite Abelian groups with isomorphic group algebras*, Duke Math. J. vol. 23 (1956) pp. 35–40.

4. M. Hall, *Group rings and extensions*, Ann. of Math. vol. 38 (1938) pp. 220–234.

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A RING ADMITTING MODULES OF LIMITED DIMENSION

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Let K be a ring with unit. A module[1] M over K is said to be *finite dimensional* if it (i) is finitely based, and (ii) contains no infinite independent set. For such a module there must exist [1, Theorem 7, p. 245] an integer n such that all bases have length n (the *invariant basis number* property), and no independent set has length greater than n. It was shown in a recent paper [1, Theorem 6, p. 245] that this property carries downward with decreasing length of basis. That is: *If K admits a module of finite dimension n, then every module over K having a basis of length $\leqq n$ is also finite dimensional.*

It was remarked (in [1]) that this leaves open the possibility that a ring could exist admitting only modules of limited dimension. That is, for some fixed integer n there might exist a ring K such that a module over K is finite dimensional if and only if it has a basis of length $\leqq n$. It is the purpose of this paper to construct such a ring for arbitrary n.

Let R be the ring of (noncommutative) polynomials generated over the field of integers modulo 2 by a countably infinite set of symbols $\{x_i, y_j\}$, with $i=1, \cdots, m=(n+2)(n+1); j=1, 2, \cdots$, where n is the fixed integer chosen. Let R' be the subring of R generated by the $\{x_i\}$. It is desired to order a (suitably restricted) set of n-dimensional row vectors of members of R'. Begin by ordering the set of all

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[1] Throughout this paper "module" will mean "left module."