AN INTERSECTION PROPERTY FOR CONES IN A LINEAR SPACE

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1. **Introduction.** The author proved in a previous paper [**2**] that if X is a real linear space with a very rudimentary topology and if $\mathfrak{F}$ is the family of all translates of a set C with an extreme point such that the intersection of any two sets in $\mathfrak{F}$ is another set in $\mathfrak{F}$ then C is a convex cone. A partial converse of that theorem is given in this note. It is shown that if C is a cone and if $x, y \in X$ exist such that $(x+C)\cap(y+C)$ is a cone K, then there exists a $z\in X$ such that $K=z+C$. In §3 examples of cones C having this property are discussed.

Let X be a linear space with real scalars. A straight line through the origin θ is defined as the set of all elements $\{\gamma x\}$, $-\infty<\gamma<\infty$ where x is any point of the set $x\neq\theta$. Straight lines not through the origin will be defined as translates of lines through θ. Alternatively, a line joining x and y in X, $x\neq y$ can be defined as the set $\{\alpha x+(1-\alpha)y \mid -\infty<\alpha<\infty\}$. The *ray* from x through y is the set $\{\alpha y+(1-\alpha)x \mid \alpha\geqq 0\}$. The *segment* joining x and y is the set $\{\alpha y+(1-\alpha)x \mid 0\leqq\alpha\leqq 1\}$. A subset $A\subset X$ is *linearly closed* in X [**2**] if every line through a point of A intersects A in a line, ray, segment or a single point. Complements of such sets are *linearly open*. A *hyperplane* in X is a maximal proper linearly closed subspace of X. A *cone* C in X with vertex θ is a set such that if $x\in C$, $\lambda x\in C$ for all $\lambda\geqq 0$. We shall make extensive use of this geometric terminology and open and closed sets in X refer to the quasi topology which has been defined above.

2. **The principal theorem.**

THEOREM 1. *Let C be a closed convex cone with vertex θ in a linear space X. Let $x, y\in X$, $x\neq y$ have the property that there exists a $z\in X$ and a cone K with vertex at the origin such that $(x+C)\cap(y+C)$ $=z+K$. Then $K=C$.*

PROOF. By the homogeneous structure of X it may be assumed that $x=\theta$ and that $C\cap(y+C)=z+K$. Evidently $C\cap(y+C)$ is closed

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and convex. Also by the convexity of C, since $z \in C$ and $z \in y+C$, $z + C \subset C \cap (y + C) = z + K$. Thus it must be shown that $z+C \supset C \cap (y+C) = z+K$. Assume that $x' \in X$ exists with $x' \in z+K$, $x' \notin z+C$. Then there exists an open set N with $x' \in N$, $N \cap (z+C)$ void since $z+C$ is closed and $x' \notin z+C$. Since C is a cone, for any $\alpha > 0$, $\alpha C = C$ and $\alpha y + C = \alpha y + \alpha C = \alpha(y+C)$. Thus $C \cap (\alpha y + C) = (\alpha C) \cap \alpha(y + C)$. However, by an elementary argument, $(\alpha C) \cap \alpha(y + C) = \alpha[C \cap (y + C)] = \alpha(z + K)$ and $\alpha(z + K) = \alpha z + \alpha K = \alpha z + K$ since K is a cone and $K = \alpha K$ for $\alpha > 0$. Thus $C \cap (\alpha y + C) = \alpha z + K$ and $C \cap (\alpha y + C)$ is a cone for any $\alpha > 0$. Let $N_\theta = N - x'$ and let $\alpha > 0$ be chosen sufficiently small to insure that $\alpha z \in N_\theta$, $\alpha(y-z) \in N_\theta$. Since $x' \in z+K$, $x'' = x' + (\alpha - 1)z \in (\alpha - 1)z + (z+K) = \alpha z + K$. Also

$$(x'' + N_\theta) \cap (\alpha z + C) = (x' + (\alpha - 1)z + N_\theta) \cap (z + (\alpha - 1)z + C)$$

is void since the sets $x' + N_\theta$ and $z+C$ have a void intersection by the choice of $N = x' + N_\theta$. Since $C \cap (\alpha y + C) = \alpha z + K$, $x'' \in C \cap (\alpha y + C)$ and $x'' \in C$, $x'' \in \alpha y + C$. However, for any point $u \in C$, $(u + N_\theta) \cap (\alpha z + C)$ is nonvoid since $\alpha z + u \in \alpha z + C$ and $(\alpha z + u) - u = \alpha z \in N_\theta$. Thus $(\alpha z + u) \in u + N_\theta$ and $\alpha z + u \in (u + N_\theta) \cap (\alpha z + C)$. Since $x'' \in \alpha z + K \subset C$, $(x'' + N_\theta) \cap (\alpha z + C)$ is nonvoid and since $(x'' + N_\theta) \cap (\alpha z + C) = [x' + (\alpha - 1)z + N_\theta] \cap [z + (\alpha - 1)z + C]$, $(x' + N_\theta) \cap (z+C) = N \cap (z+C)$ is nonvoid contrary to assumption. Thus $x' \in z+C$ and $z+K \subset z+C$. Hence $K = C$.

A C-cone [1] is a cone having the property that for any $x, y \in X$, there exists $z \in X$ with $(x+C) \cap (y+C) = z+C$. Thus the previous theorem gives rise to the following corollary.

Corollary. *If C is a closed cone such that for any two elements $x, y \in X$ $(x+C) \cap (y+C)$ is a cone, then C is a C-cone.*

Theorem 1 also yields slight generalizations of the characterization theorems for L and C spaces given in [3] and [1]. In these theorems the hypotheses that the cones which characterize the spaces be C-cones can be replaced by the slightly less restrictive hypotheses of the preceding corollary.

3. Examples of cones with property T.

Definition. A cone C with vertex θ has property T if there exists $x \notin C \cup (-C)$ and $y \in X$ such that $C \cap (x+C) = y+C$.

Certain cones having property T are investigated in this section and an example of the use of such a cone in the structure of function spaces is given. A more complete investigation of the structure of such cones and their applications is planned in a subsequent paper.

THEOREM 2. *Let H be a hyperplane in X and let C_H be any closed convex cone in H with vertex θ. Let $x \in X \backslash H$ and let r be the ray from θ through x. Let C be the convex set determined by r and C_H. Then C is a closed convex cone such that if $y \in C_H$, $y \neq \theta$, and L is the two space determined by θ, x, y, then for any $u \in L$ there exists $z \in L$ with $C \cap (u+C) = z+C$.*

PROOF. If $u \in C \cup (-C)$ the conclusion is obvious and $z=u$ or $z=\theta$. Assume $u \notin C \cup (-C)$ and let r' be the ray from θ through y. By elementary arguments it can be seen that either $(u+r) \cap r'$ or $(u+r') \cap r$ is nonvoid. By making a translation of the vertex and an interchange of C and $u+C$ if necessary, it can always be assumed that $(u+r') \cap r$ is nonvoid. Let $z=(u+r') \cap r$. Then $z \in C \cap (u+C)$. It can easily be established by elementary arguments that if r'' is a ray lying entirely in C then any translate of r'' by a point of C also lies in C. Making use of this fact, we show that $C \cap (u+C)$ is a cone and hence by Theorem 1, a translate of C. Let $w \in C \cap (u+C)$. By construction of C this implies the existence of real numbers α, $0 \leqq \alpha \leqq 1$ and $\gamma > 1$ such that $w = \alpha v + (1-\alpha) v'$ where $v = \gamma z$ and $v' \in C_H$. If it can be shown that $w - z \in C$ then the ray from θ through $w - z \in C$ and hence the ray from z through w is in C. However, $w = \alpha\gamma z + (1-\alpha)v'$, $w - z = \alpha\gamma z + (1-\alpha)v' - z = (\alpha\gamma - 1)z + (1-\alpha)v'$. Let $v'' = (1/\gamma)v + (1-1/\gamma)v' = z + (1-1/\gamma)v'$. Then $v'' \in C$ and $v'' \in z+H = u+H$. Since each component of $C \backslash (z+H)$ is convex, the half open segment, $\{\beta v' + (1-\beta)v'' \mid 0 < \beta \leqq 1\}$ lies entirely in the component of $C \backslash (z+H)$ which contains C_H and hence no point of this set lies in $C \cap (u+C)$. Thus since $w \in C \cap (u+C)$ and since the points of the segment from v to v' with $\alpha < 1/\gamma$ lie between v'' and v', $\alpha \geqq 1/\gamma$. Hence $\alpha\gamma - 1 \geqq 0$. Thus since z, $v' \in C$, $w - z \in C$ and the entire ray from θ through $w - z$ is in C. Thus the ray from z through w lies in C. A similar argument shows that the ray from z through w also lies in $u+C$. Thus for any point $w \in C \cap (u+C)$ the ray from z through w lies in $C \cap (u+C)$. This shows that $C \cap (u+C)$ is a closed convex cone with vertex z. By Theorem 1, $C \cap (u+C) = z+C$.

THEOREM 3. *Let X, H, C_H be defined as in Theorem 2. Let $x, y \in X \backslash H$ be chosen in such a way that the line segment joining x and y intersects C_H in a point distinct from θ. Let r_x, r_y be rays from θ through x and y respectively and let C be the cone which is the convex hull of r_x, r_y, C_H. Let $L(x, y)$ be the two dimensional subspace of X determined by θ, x, y and let $u \in L(x, y)$. Then there exists $z \in L(x, y)$ such that $C \cap (u+C) = z+C$.*

PROOF. If $u \in C \cup (-C)$ then the theorem is obvious and $z=u$ or $z=\theta$. It can thus be assumed with no loss of generality that u, x are in the same component of $X\backslash H$. Thus $r_x\cap(u+r_y)$ is nonvoid by construction. Let $z=r_x\cap(u+r_y)$. It will be shown as in the previous theorem that $C\cap(u+C)$ is a cone with vertex at z and is hence equal to $z+C$. Let $w\in C\cap(u+C)$. It must be shown that the ray from z through w is in $C\cap(u+C)$. If $w\in z+H$ or $z+H$ separates w from H, the same procedure as was used in Theorem 2 shows that $w-z\in C$. Since $u+C$ is a translate of C, $w-z+u\in u+C$ and the rays from θ, u through $w-z$, $w-z+u$ respectively lie in C and $u+C$. Thus the ray from z through w also lies in C and $u+C$ and hence in $C\cap(u+C)$. If w lies on the other side of $z+H$, a similar argument can be applied to $u+C$ and C and the ray from z through w lies in $C\cap(u+C)$. Thus $C\cap(u+C)$ is a cone with vertex z and by Theorem 1, $C\cap(u+C)$ $=z+C$.

COROLLARY. *Let a cone C be the closure of the convex set determined by a set of rays $\{r_\alpha\}$ starting from θ and assume that these rays have the following properties.* (a) *the linear space spanned by $\{r_\alpha\}$ is dense in X,* (b) *if $r_{\alpha_0}\in\{r_\alpha\}$ there exists a second ray $r'_{\alpha_0}\in\{r_\alpha\}$ such that the remainder of the rays in $\{r_\alpha\}$ generates a linear space whose closure is a hyperplane H_{α_0} which separates r_α and r'_α and such that the two dimensional subspace determined by r_{α_0} and r'_{α_0} intersects $C_{H_\alpha}=H_\alpha\cap C$. Then if u is any point in the two space determined by any r_{α_0} and r'_{α_0}, there exists a $z\in X$ such that $C\cap(u+C)=z+C$.*

EXAMPLE. Let l be the space of all real summable sequences. Let z_i^+ be the point $(0, 0, \cdots, 0, 1, 0, \cdots)$, $i=1, 2, 3, \cdots$ and z_i^- $=(0, 0, \cdots, 0, -1, 0, \cdots)$, $i=1, 2, \cdots$ with 0 everywhere except in the ith place. These points are then extreme points of the unit sphere in l. Let C_i^+ be the cone with vertex z_i^+ which is generated by the unit sphere. This is easily seen to be convex and closed in the norm topology. Then C_i^+ satisfies the conditions of the preceding corollary since evidently if r_j^+, r_j^- are the rays from z_i^+ through z_j^+, z_j^- respectively and H_j is the closed subspace determined by the rest of the $r_k^\pm$, H_j is a hyperplane and r_j^+, r_j^- satisfy the conditions set forth there. Similarly for C_i^- for each i. Thus the unit sphere in l is equal to $C_i^+\cap C_i^-$ for any $i=1, 2, \cdots$ where the C_i^+, C_i^- satisfy the conditions of the preceding corollary.

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