A REMARK CONCERNING A THEOREM OF B. FRIEDMAN

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In [1], the following theorem is stated: let T be a densely defined linear operator with closed range in the Hilbert space $\mathcal{H}$, with a densely defined adjoint T^* also having closed range. Let ϕ, ψ be vectors in $\mathcal{H}$, and let $\phi\otimes\psi$ be the operator defined by $\phi\otimes\psi(x) = (x, \phi)\psi$. Then $T+\phi\otimes\psi$ also has closed range.

Of course, the fact that T^* is densely defined implies that T is pre-closed; but an examination of the proof shows that it actually requires that T be a *closed* operator. Under this assumption, a simpler proof can be given; and the need for some such condition will be shown by example.

THEOREM. *Let T be a closed, densely defined operator with closed range. Then $S=T+\phi\otimes\psi$ also has a closed range.*

PROOF. The nullspace $\mathfrak{N}_T$ of a closed operator T is closed, and its domain $\mathfrak{D}_T$ is the sum of the two subspaces $\mathfrak{N}_T$ and $\mathfrak{D}_T\cap\mathfrak{N}_T^{\perp}=\mathcal{S}_T$, since x in $\mathfrak{D}_T$ can be written $(x-P_{\mathfrak{N}_T}x)+P_{\mathfrak{N}_T}x$ (where $P_{\mathfrak{M}}$ denotes the orthogonal projection on the subspace $\mathfrak{M}$). If we use the graph norm on $\mathfrak{D}_T$, given by the inner product $\langle x, y\rangle = (x, y)+(Tx, Ty)$, then $\mathfrak{N}_T$ and $\mathcal{S}_T$ are complete, and $\mathfrak{D}_T$ is their Hilbert space direct sum.

T restricted to $\mathcal{S}_T$ is a 1-1 continuous operator from $\mathcal{S}_T$ (in the graph norm) to the range $\mathfrak{R}_T$ of T. The closed graph theorem tells us that its inverse R is continuous, as an operator from the Hilbert space $\mathfrak{R}_T$ to $\mathcal{S}_T$. Now, the orthogonal complement $[\phi]^{\perp}$ of ϕ in $\mathcal{H}$ is closed in $\mathcal{H}$. Thus its intersection with $\mathcal{S}_T$ is closed in the graph norm. Then $R^{-1}([\phi]^{\perp}\cap\mathcal{S}_T)=T([\phi]^{\perp}\cap\mathcal{S}_T)$ is closed in $\mathcal{H}$. $T([\phi]^{\perp}\cap\mathcal{S}_T)=S([\phi]^{\perp}\cap\mathcal{S}_T)$ $\subset\mathfrak{R}_S\subset\mathfrak{R}_T+[\psi]=T(\mathcal{S}_T)+[\psi]$. Now, the codimension of $T([\phi]^{\perp}\cap\mathcal{S}_T)$ in $T(\mathcal{S}_T)+[\psi]$ is at most two, so that of $T([\phi]^{\perp}\cap\mathcal{S}_T)$ in $\mathfrak{R}_s$ is again at most two. Since $T([\phi]^{\perp}\cap\mathcal{S}_T)$ is closed, $\mathfrak{R}_s$ is also closed.

REMARK 1. If T had been merely preclosed, but with closed range, then it is easy to see $\mathfrak{R}_{\overline{T}}=\mathfrak{R}_T$, so that $\overline{S}=\overline{T}+\phi\otimes\psi$ has closed range.

REMARK 2. Here is an example of an operator T which is densely defined, bounded, and has closed range, and whose adjoint T^* is therefore bounded and has closed range, but for which $S=T+\phi\otimes\psi$ will *not* have closed range, for certain ϕ and ψ.

Let $\mathcal{H}_0$ be a proper infinite-dimensional subspace of $\mathcal{H}$, and

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$\psi, \psi_1, \psi_2, \psi_3, \cdots$ an orthonormal basis for $\mathcal{H}_0$. Let $\psi_n' = n^{-1/2}((n-1)^{1/2}\psi + \psi_n)$. Then the set $\Psi = \{\psi, \psi_1', \psi_2', \cdots\}$ is linearly independent, and $\|\psi_n' - \psi\|^2 \to 0$. Enlarge Ψ to a maximal linearly independent set Φ in $\mathcal{H}_0$. Thus, the linear combinations of elements of Φ span $\mathcal{H}_0$. Let $\mathcal{K}_0$ be the set of all linear combinations of elements of $\Phi - \{\psi\}$. Then $\mathcal{K}_0$ has the following properties:

(1) $\mathcal{K}_0$ is dense in $\mathcal{H}_0$.

(2) $\psi \notin \mathcal{K}_0$.

(3) $\mathcal{K}_0 + [\psi] = \mathcal{H}_0$.

Let ϕ be any unit vector in $\mathcal{H}_0^\perp$. Let T be the restriction of $P_{\mathcal{H}_0} - \phi \otimes \psi$ to $\mathcal{K}_0 + \mathcal{H}_0^\perp$. Then clearly $\mathcal{R}_T \subset \mathcal{H}_0$. Further, $T | \mathcal{K}_0 = P_{\mathcal{H}_0} | \mathcal{K}_0$, so $\mathcal{K}_0 \subset \mathcal{R}_T$. Finally, $T\phi = -\psi$, so $\mathcal{R}_T = \mathcal{H}_0$. Notice also that $T^* = P_{\mathcal{H}_0} - \psi \otimes \phi$, and so if $x = x_1 + x_2 + \alpha\psi$, where $x_1 \in \mathcal{H}_0 \cap [\psi]^\perp$, $x_2 \in \mathcal{H}_0^\perp$, then $T^*x = x_1 + \alpha(\psi - \phi)$. Thus $\mathcal{R}_{T^*} = (\mathcal{H}_0 \cap [\psi]^\perp) + [\psi - \phi]$, clearly closed. However, $T + \phi \otimes \psi = P_{\mathcal{H}_0} | \mathcal{K}_0$ has $\mathcal{K}_0$ as its range.

Reference

1. B. Friedman, *Operations with a closed range*, Comm. Pure Appl. Math. VIII, vol. 4 (1955) p. 539.

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